

# FRAMED SYMPLECTIC SHEAVES ON SURFACES

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**ABSTRACT.** A framed symplectic sheaf on a smooth projective surface  $X$  is a torsion-free sheaf  $E$  together with a trivialization on a divisor  $D \subseteq X$  and a morphism  $\Lambda^2 E \rightarrow \mathcal{O}_X$  satisfying some additional conditions. We construct a moduli space for framed symplectic sheaves on a surface, and present a detailed study for  $X = \mathbb{P}_{\mathbb{C}}^2$ . In this case, the moduli space is irreducible and admits an ADHM-type description and a birational proper map onto the space of framed symplectic ideal instantons.

## 1. INTRODUCTION

Let  $l$  be a fixed line in the complex projective plane  $\mathbb{P}_{\mathbb{C}}^2$ . An  $l$ -framed sheaf of charge  $n$  is defined to be a pair  $(E, a)$  where  $E$  is a torsion-free coherent sheaf on  $\mathbb{P}_{\mathbb{C}}^2$  of generic rank  $r$  and  $c_2(E) = n$ , and  $a : E|_l \rightarrow \mathcal{O}_l^{\oplus r}$  is an isomorphism, which we call *framing*. Framed sheaves are parameterized by a fine moduli space, usually denoted  $\mathcal{M}(r, n)$ . The interest in this space arises from Gauge theory. An  $SU(r)$ -instanton on the four sphere with charge  $n$  is a principal  $SU(r)$ -bundle endowed with an anti-selfdual connection. As explained in [ADHM], instantons can be described by means of linear algebraic data. The upshot here is an interpretation of the moduli space of instantons as an algebro-geometric object, namely as the moduli space  $\mathcal{M}^{reg}(r, n)$  of holomorphic vector bundles on the projective plane with a framing on a line, see [Do]. There exists a partial compactification  $\mathcal{M}^{reg}(r, n) \subseteq \mathcal{M}_0(r, n)$ ; this is obtained allowing instantons to degenerate to a so-called *ideal instanton*, that is, a bundle with a connection whose square curvature density has distributional degenerations at a finite number of points. This space, which is also called *Uhlenbeck space*, can be thought of as an affine  $\mathbb{C}$ -scheme. It is very singular, and it does not possess very well defined modular properties; however, there exists a natural map

$$\pi : \mathcal{M}(r, n) \rightarrow \mathcal{M}_0(r, n)$$

which is in fact a resolution of singularities. Both  $\mathcal{M}_0(r, n)$  and  $\mathcal{M}(r, n)$  contain an open subscheme isomorphic to  $\mathcal{M}^{reg}(r, n)$ , over which  $\pi$  is an isomorphism.

If we change  $SU(r)$  for another real simple Lie group  $G$ , we have  $G$ -analogues of  $\mathcal{M}^{reg}(r, n)$  (we can take the moduli space of framed principal  $G_{\mathbb{C}}$ -bundles) and also for  $\mathcal{M}_0(r, n)$  (see for example [Bal, BFG, Ch, NS]), but this time no modular desingularization of the Uhlenbeck space is known. Indeed, it is not clear what should be the correct definition of “weak principal

bundle" to use in order to obtain a smooth partial compactification, as one does with torsion-free sheaves in the classical case.

In this paper we focus on the case of the symplectic group  $G = SP_r$ . We define a *framed symplectic sheaf* to be a framed sheaf  $(E, a)$  with a morphism  $\varphi : \Lambda^2 E \rightarrow \mathcal{O}_{\mathbb{P}^2}$  which behaves as a symplectic form on the locally free locus of  $E$ . The main point of this paper is to prove that framed symplectic sheaves admit a fine module space as well, denoted  $\mathcal{M}_\Omega(r, n)$ , which comes with a proper birational morphism onto the symplectic variant of the Uhlenbeck space.

The paper is organized as follows. Section 2 fixes some notation and introduces the definition of the moduli functor for framed symplectic sheaves. Moreover, some lemmas for later use are stated. Section 3 describes the construction of moduli space for a general smooth projective surface  $X$ , where we fix a suitable framing divisor  $D \subseteq X$ , and provides the computation of the tangent space at a point of the moduli space. Section 4 specializes to the case  $(X, D) = (\mathbb{P}^2, l)$ , and presents an alternative definition of the moduli space by means of linear data, in the spirit of the ADHM construction for the classical case. In section 5 this description is applied in order to prove irreducibility of the moduli space on  $\mathbb{P}^2$ . Section 6 deals with Uhlenbeck spaces, and provides some results on the singularities of  $\mathcal{M}_\Omega(r, n)$ .

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## 2. PRELIMINARIES AND NOTATION

**2.1. Bilinear forms on sheaves.** Let  $X$  be a  $\mathbb{K}$ -scheme with  $\mathbb{K}$  an algebraically closed field with  $\text{char}(\mathbb{K}) \neq 2$ , and let  $E$  be an  $\mathcal{O}_X$ -module. The skew-symmetric square  $\Lambda^2 E$  and the symmetric square  $S^2 E$  of  $E$  are defined as the sheafifications of the presheaves

$$U \mapsto S^2 E(U), \quad U \mapsto \Lambda^2 E(U).$$

In particular, they can be naturally written as quotients of  $E^{\otimes 2}$ . Let  $i \in \text{Aut}(E^{\otimes 2})$  be the natural switch morphism (on stalks:  $i(e \otimes f) = f \otimes e$ ). Let  $G$  be another  $\mathcal{O}_X$ -module and  $\varphi : E^{\otimes 2} \rightarrow G$  a morphism. We call  $\varphi$  symmetric (resp. skew-symmetric) if  $\varphi \circ i = \varphi$  (resp.  $\varphi \circ i = -\varphi$ ).  $S^2 E$  and  $\Lambda^2 E$  satisfy the obvious universal properties

$$\begin{array}{ccc} E^{\otimes 2} & \xrightarrow{\forall \text{symm}} & G \\ \downarrow & \nearrow \exists! & \\ S^2 E & & \end{array} \quad \begin{array}{ccc} E^{\otimes 2} & \xrightarrow{\forall \text{skew}} & G \\ \downarrow & \nearrow \exists! & \\ \Lambda^2 E & & \end{array}$$

and in fact fit into a split-exact sequence

$$0 \rightarrow S^2 E \rightarrow E^{\otimes 2} \rightarrow \Lambda^2 E \rightarrow 0.$$

*Remark 2.1.* A bilinear form  $\varphi \in \text{Hom}(E^{\otimes 2}, G)$  naturally corresponds to an element of

$$\text{Hom}(E, G \otimes E^\vee) \cong \text{Hom}(E^{\otimes 2}, G).$$

If  $\varphi$  is symmetric or skew, it corresponds to a unique form in  $\text{Hom}(S^2 E, G)$  or  $\text{Hom}(\Lambda^2 E, G)$ . We shall make a systematic abuse of notation by calling  $\varphi$  all these morphisms.

The modules  $S^2 E$  and  $\Lambda^2 E$  are coherent or locally free if  $E$  is, and are well-behaved with respect to pullbacks, meaning that for a given morphism of schemes  $f : Y \rightarrow X$  one has natural isomorphisms

$$S^2 f^* E \cong f^* S^2 E, \quad \Lambda^2 f^* E \cong f^* \Lambda^2 E.$$

The following lemmas will be useful later.

**Lemma 2.2.** *Let  $K \rightarrow H \rightarrow E \rightarrow 0$  be an exact sequence of coherent  $\mathcal{O}_X$ -modules. There is an exact sequence*

$$K \otimes H \rightarrow \Lambda^2 H \rightarrow \Lambda^2 E \rightarrow 0.$$

*If furthermore  $H$  is locally free and  $K \rightarrow H$  is injective, define  $K \wedge H$  to be the image of the subsheaf  $K \otimes H \subseteq H^{\otimes 2}$  under  $H^{\otimes 2} \rightarrow \Lambda^2 H$ . Then there is a natural isomorphism*

$$K \wedge H \cong K \otimes H / (K \otimes H \cap S^2 H)$$

*and thus an exact sequence*

$$0 \rightarrow K \wedge H \rightarrow \Lambda^2 H \rightarrow \Lambda^2 E \rightarrow 0.$$

**Lemma 2.3.** [GS2, Lemma 0.9] *Let  $Y$  be a scheme and  $h : \mathcal{F} \rightarrow \mathcal{G}$  a morphism of coherent sheaves on  $X \times Y$ . Assume  $\mathcal{G}$  is flat over  $Y$ . There exists a closed subscheme  $Z \subseteq Y$  such that the following universal property is satisfied: for any  $s : S \rightarrow Y$  with  $(1_X \times s)^* h = 0$ , one has a unique factorization of  $s$  as  $S \rightarrow Z \rightarrow Y$ .*

**Lemma 2.4.** [Mc, Thm 12.1] *Let  $R \rightarrow S$  be a morphism of unitary commutative rings, let  $N$  be an  $S$ -module and  $N'$  be an  $R$ -module. There exists a spectral sequence with  $E_{p,q}^2 = \text{Ext}_S^q(\text{Tor}_p^R(S, N'), N)$  converging to  $\text{Ext}_R(N', N)$ . In particular, if  $N'$  is  $R$ -flat, we get an isomorphism*

$$\text{Ext}_S^q(S \otimes N', N) \cong \text{Ext}_R^q(N', N).$$

**Lemma 2.5.** *Suppose  $f : X \rightarrow Y$  is a morphism of  $\mathbb{K}$ -schemes of finite type. Assume that the following conditions hold:*

- (1) *the map of sets  $f(\mathbb{K}) : X(\mathbb{K}) \rightarrow Y(\mathbb{K})$  is injective;*
- (2) *for any closed point  $x \in X$  the linear map  $Tf_x : T_x X \rightarrow T_{f(x)} Y$  is a monomorphism;*
- (3)  *$f$  is proper.*

*Then  $f$  is a closed embedding.*

**2.2. Framed symplectic sheaves.** Let  $X$  be a projective surface,  $D \subseteq X$  a big and nef divisor and  $W$  a finite-dimensional vector space.

**Definition 2.6.** A framed sheaf on  $X$  is a pair  $(E, \alpha)$  where  $E$  is a torsion-free sheaf of rank  $r$  on  $X$  and  $\alpha : E_D \rightarrow \mathcal{O}_D \otimes W$  is an isomorphism. A morphism of framed sheaves  $(E, \alpha) \rightarrow (E', \alpha')$  is a morphism of sheaves  $f : E \rightarrow E'$  where  $\alpha' \circ f_D = \lambda \circ \alpha$  for a nonzero scalar  $\lambda \in \mathbb{K}$ .

If  $(E, \alpha)$  is a framed sheaf,  $E$  is locally free in a neighborhood of  $D$ .

Framed sheaves are easily defined in families. An  $S$ -family of framed sheaves on  $X$  for a given scheme  $S$  consists of a pair  $(\mathcal{E}, \alpha)$  where  $\mathcal{E}$  is a  $S$ -flat sheaf on  $X_S$  and  $\alpha$  is an isomorphism

$$\alpha : \mathcal{E}_{D_S} \rightarrow D_S \otimes W.$$

The pullback family via a morphism  $S \rightarrow S'$  is well defined. We obtain a functor

$$\mathfrak{M}_X^D(r, n) : Sch^{op} \rightarrow Set$$

assigning to  $S$  the set of isomorphism classes of  $S$ -families of framed sheaves.

Fix now some symplectic form  $\Omega : \Lambda^2 W \rightarrow \mathbb{K}$  (this forces  $r = \dim(W)$  to be even). This yields a symplectic form on  $\mathcal{O}_D \otimes W$ , still denoted by  $\Omega$ .

**Definition 2.7.** A framed symplectic sheaf is a triple  $(E, a, \varphi)$  where  $(E, a)$  is a framed sheaf with  $\det(E) \cong \mathcal{O}_X$  and  $\varphi : \Lambda^2 E \rightarrow \mathcal{O}_X$  is a morphism satisfying

$$(2.1) \quad \varphi_D = \Omega \circ \Lambda^2 a.$$

*Remark 2.8.* The morphism  $\varphi : E \rightarrow E^\vee$  is an isomorphism once restricted to  $D$ ; consequently, the same holds on an open neighborhood of  $D$ . As  $\det(E) \cong \det(E^\vee) \cong \mathcal{O}_X$ , we obtain  $c_1(E) = 0$ . Furthermore, we deduce that  $\det(\varphi)$  is a nonzero constant. It follows that  $\varphi$  induces an isomorphism  $E_U \rightarrow E_U^\vee$  on the entire open 2-codimensional set  $X \setminus \text{sing}(E) = U \supset D$ .

If  $(E, a)$  is a framed sheaf with  $c_1(E) = 0$  and we are given a morphism  $\varphi : \Lambda^2 E \rightarrow \mathcal{O}_X$  satisfying the compatibility condition 2.1, we get in fact  $\det(E) \cong \mathcal{O}_X$ . Indeed, since  $\varphi$  is an isomorphism in a neighborhood of  $D$ , we obtain an exact sequence

$$0 \rightarrow E \rightarrow E^\vee \rightarrow \text{coker}(\varphi) \rightarrow 0.$$

The sheaf  $\text{coker}(\varphi)$  must be supported on a subscheme of dimension strictly smaller than 2, and  $c_1(\text{coker}(\varphi)) = 0$  forces

$$\dim(\text{coker}(\varphi)) = 0.$$

This implies that  $\varphi$  must be an isomorphism outside a zero-dimensional subscheme of  $X$ , from which

$$\det(E) \cong \det(E^\vee)$$

follows. Finally, consider the dual map

$$\varphi^\vee : E^{\vee\vee} \rightarrow E^\vee;$$

it is a skew-symmetric isomorphism of vector bundles, i.e. a symplectic form. We may conclude  $\det(E^\vee) \cong \mathcal{O}_X$ . From now on, we will always omit the hypothesis on the determinant while working with framed symplectic sheaves, as we will consider exclusively framed sheaves whose first Chern class vanishes. Thus, the discrete invariants of framed symplectic sheaves will be the positive integer  $r$  and the nonnegative integer  $c_2$ .

We define a morphism of framed symplectic sheaves  $(E, \alpha, \varphi) \rightarrow (E', \alpha', \varphi')$  to be a morphism  $f$  of framed sheaves where  $\varphi' \circ \Lambda^2 f = \lambda \varphi$  for a (again nonzero)  $\lambda \in \mathbb{K}$ .

*Remark 2.9.* We shall soon see (Prop. 3.1) that, for a given pair of framed sheaves  $(E, a)$  and  $(F, b)$  with the same invariants, there is at most one isomorphism  $f : E \rightarrow F$  satisfying

$$a = b \circ f_D.$$

This implies that  $(E, a)$  can support *at most one* structure of framed symplectic sheaf, in the following sense. If  $\varphi : \Lambda^2 E \rightarrow \mathcal{O}_X$  is a symplectic form, it induces an isomorphism  $\varphi^\vee : E^{\vee\vee} \rightarrow E^\vee$ . Furthermore,  $E^{\vee\vee}$  and  $E^\vee$  inherit framings from  $(E, a)$ , and  $\varphi^\vee$  preserves these framings, as  $\varphi$  is  $\Omega$ -compatible with  $a$ . As a consequence, any other symplectic form  $\varphi'$  on  $E$  satisfies  $(\varphi')^\vee = \varphi^\vee$ , from which  $\varphi = \varphi'$  follows. In particular:

**Lemma 2.10.** *If  $(E, \alpha, \varphi)$  is a framed symplectic sheaf,  $\text{Hom}(\Lambda^2 E, \mathcal{O}_X(-D)) = 0$  (i.e., the symplectic form has no nontrivial infinitesimal automorphisms).*

*Proof.* Let  $\psi : \Lambda^2 E \rightarrow \mathcal{O}_X(-D)$  be a morphism. By means of the exact sequence

$$0 \rightarrow \mathcal{O}_X(-D) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_D \rightarrow 0$$

we can think of  $\psi$  as a morphism  $\Lambda^2 E \rightarrow \mathcal{O}_X$  which vanishes on  $D$ . Now, for a nonzero scalar  $\lambda$  consider  $\psi_\lambda = \psi + \lambda\varphi$ . If we choose a square root of  $\lambda^{1/2}$ , we obtain that  $(E, \lambda^{1/2}\alpha, \psi_\lambda)$  is a framed symplectic sheaf, but also  $(E, \lambda^{1/2}\alpha, \lambda\varphi)$  is. This forces  $\psi_\lambda = \lambda\varphi$ , i.e.  $\psi = 0$ .  $\square$

We can define framed symplectic sheaves in families again; an  $S$ -family of framed symplectic sheaves will be a triple  $(\mathcal{E}, \alpha, \Phi)$  with  $(\mathcal{E}, \alpha)$  an  $S$ -family of framed sheaves, and  $\Phi : \Lambda^2 \mathcal{E} \rightarrow \mathcal{O}_{X_S}$  a morphism such that  $\Phi|_{D_S} = \Omega \circ \Lambda^2 \alpha$ . The corresponding functor will be denoted  $\mathfrak{M}_{X, \Omega}^D(r, n)$ .

### 3. MODULI SPACES OF FRAMED SYMPLECTIC SHEAVES ON SURFACES

**3.1. Framed sheaves as Huybrechts-Lehn framed pairs.** It is possible to construct a fine moduli space for the functor  $\mathfrak{M}_X^D(r, n)$ . Let  $F$  be a fixed sheaf on  $X$ . A framed module is a pair  $(E, a)$  where  $E$  is a coherent sheaf on  $X$ , and  $a : E \rightarrow F$  is a morphism. A framed sheaf  $(E, a)$  is a special example of framed module, with  $F = \mathcal{O}_D \otimes W$ ,  $E$  torsion-free and  $a$  inducing an isomorphism once restricted to  $D$ . In [HL, Def. 1.1 and Thm 2.1], a (semi)stability condition depending on a numerical polynomial  $\delta$  and on a fixed polarization  $H$  is defined, and a boundedness result is provided for framed modules.

Let  $c \in H^*(X, \mathbb{Q})$ . In [BM, Thm 3.1], it is shown that there exist a polarization  $H$  and a numerical polynomial  $\delta$  such that any framed sheaf  $(E, a)$  with Chern character  $c(E) = c$  is  $\delta$ -stable as a framed module. This is a crucial step in order to realize the moduli space of framed sheaves  $\mathcal{M}_X^D(r, n)$  as an open subscheme of the moduli space of  $\delta$ -semistable framed modules as defined in [HL].

On the other hand, in [GS1] the authors present the construction of a coarse moduli space for semistable symplectic sheaves. A symplectic sheaf is a pair  $(E, \varphi)$  where  $E$  is a coherent torsion-free sheaf on  $X$  and  $\varphi : \Lambda^2 E \rightarrow \mathcal{O}_X$  is a morphism inducing a symplectic form on the maximal open subset of  $X$  over which  $E$  is locally free. We will construct a fine moduli space for  $\mathfrak{M}_{X, \Omega}^D(r, n)$  by means of a blend of the previous two constructions. Fix two integers  $r > 0$  and  $n \geq 0$ , let  $c \in H^*(X, \mathbb{Q})$  be the Chern character of a sheaf  $E \in \text{Coh}(X)$  with generic rank  $r$ ,  $c_1(E) = 0$  (as we shall be working with symplectic sheaves) and  $c_2(E) = n$ , and fix  $H$  a polarization as in [BM, Thm 3.1]. Let  $P_{r, n} = P_E$  be the corresponding Hilbert polynomial. Keeping now in mind that in this setting framed sheaves are a particular type of *stable* framed modules, we obtain the following proposition as an immediate consequence of [HL, Thm. 2.1] and [HL, Lemma 1.6].

**Proposition 3.1.** *Let  $P = P_{r, n}$ .*

- (1) *There exists a positive integer  $m_0$  such that for any  $m \geq m_0$  and for any framed sheaf  $(E, a)$  on  $X$  with Hilbert polynomial  $P$ ,  $E$  is  $m$ -regular,  $H^i(X, \mathcal{O}_X(m)) = 0 \forall i > 0$ ,  $H^1(X, \mathcal{O}_D(m)) = 0$  and  $P(m) = h^0(E(m))$ .*
- (2) *Any morphism of framed sheaves  $(E, a), (F, b)$  with the same Hilbert polynomial is an isomorphism. Furthermore, there exists a unique morphism  $f : (E, a) \rightarrow (F, b)$  satisfying  $f_D \circ b = a$ , and any other morphism between them is a nonzero multiple of  $f$ .*

**3.2. Parameter spaces.** We need to recall how to endow the set of isomorphism classes of framed sheaves  $\mathcal{M}_X^D(r, n)$  with a scheme structure, which makes it a fine moduli space for the functor  $\mathfrak{M}_{X, \Omega}^D(r, n)$  (for the full details, see [HL]), as it will be useful for constructing the moduli space  $\mathcal{M}_{X, \Omega}^D(r, n)$ . The construction is standard; we obtain both moduli spaces as geometric quotients of suitable parameter spaces, defined using *Quot* schemes. The relevant parameter spaces will be defined in this subsection.

Fix a polynomial  $P = P_{r, n}$  and a positive integer  $m \gg 0$  as in Prop. 3.1, and let  $V$  be a vector space of dimension  $P(m)$ . Let  $H = V \otimes \mathcal{O}_X(-m)$ . Consider the projective scheme

$$\text{Hilb}(H, P) \times \mathbb{P}(\text{Hom}(V, H^0(\mathcal{O}_D(m)) \otimes W)^\vee) =: \text{Hilb} \times P_{fr},$$

where  $\text{Hilb}(H, P)$  is the Grothendieck *Quot* scheme parameterizing equivalence classes of quotients

$$q : H \rightarrow E, P_E = P$$

on  $X$ . Define  $Z$  as the subset of pairs  $([q : H \rightarrow E], A)$  such that the map  $H \rightarrow \mathcal{O}_D \otimes W$  induced by  $A$  factors through  $E$ :

$$\begin{array}{ccc} H & \xrightarrow{A} & \mathcal{O}_D \otimes W \\ q \downarrow & \nearrow \bar{A} & \\ E & & \end{array}$$

$Z$  can be in fact interpreted as a closed subscheme as follows. Choose a universal quotient

$$q_{Hilb} : V \otimes \mathcal{O}_{X \times Hilb} \otimes p_X^* \mathcal{O}_X(-m) =: \mathcal{H} \rightarrow \mathcal{E}.$$

The pullback of the universal map

$$V \otimes \mathcal{O}_{P_{fr}} \rightarrow H^0(\mathcal{O}_D(m) \otimes W) \otimes \mathcal{O}_{P_{fr}}$$

to  $X \times Hilb \times P_{fr}$  yields a morphism

$$\mathcal{A}_{univ} : V \otimes \mathcal{O}_{X \times Hilb \times P_{fr}} \otimes p_X^* \mathcal{O}_X(-m) \rightarrow \mathcal{O}_{D \times Hilb \times P_{fr}} \otimes W.$$

The pullback of  $q_{Hilb}$ , induces a diagram

$$\begin{array}{ccccccc} 0 \longrightarrow \mathcal{K} & \longrightarrow & V \otimes \mathcal{O}_{X \times Hilb \times P_{fr}} \otimes p_X^* \mathcal{O}_X(-m) & \longrightarrow & p_{Hilb \times X}^*(\mathcal{E}) & \longrightarrow & 0 \\ & \searrow \mathcal{A}_{univ}^\mathcal{K} & \downarrow \mathcal{A}_{univ} & & \swarrow & & \\ & & \mathcal{O}_{D \times Hilb \times P_{fr}} \otimes W & & & & \end{array}$$

The closed subscheme  $Z \subseteq Hilb \times P_{fr}$  we are looking for is the one defined by Lemma 2.3, where  $Y = Hilb \times P_{fr}$  and  $h = \mathcal{A}_{univ}^\mathcal{K}$ . We denote by  $\overset{\circ}{Z} \subseteq Z$  the open subscheme defined by requiring that the pullback  $a = \bar{A}_D : E_D \rightarrow \mathcal{O}_D \otimes W$  is a framing.

Similarly, we can take the product

$$Hilb(H, P) \times \mathbb{P}(Hom(\Lambda^2 V \rightarrow H^0(\mathcal{O}_X(2m)))^\vee) =: Hilb \times P_{symp}$$

and consider the closed subscheme  $Z' \subseteq Hilb \times P_{symp}$  given by pairs  $([q], \phi)$  such that the map  $\Lambda^2 H \rightarrow \mathcal{O}_X$  induced by  $\phi$  descends to some  $\varphi : \Lambda^2 E \rightarrow \mathcal{O}_X$ :

$$\begin{array}{ccc} \Lambda^2 H & \xrightarrow{\phi} & \mathcal{O}_X \\ \Lambda^2 q \downarrow & \nearrow \varphi & \\ \Lambda^2 E & & \end{array}$$

We define the very last closed subscheme  $Z_\Omega \subseteq Hilb \times P_{fr} \times P_{symp}$  as follows. First, consider the scheme-theoretic intersection

$$p_{Hilb \times P_{fr}}^{-1}(Z) \cap p_{Hilb \times P_{symp}}^{-1}(Z'),$$

whose closed points are triples  $([q], A, \phi)$  satisfying:  $A$  descends to  $\alpha : E \rightarrow \mathcal{O}_D$ ,  $\phi$  descends to  $\varphi : \Lambda^2 E \rightarrow \mathcal{O}_X$ . This scheme has again a closed subscheme defined by triples satisfying the

following compatibility on  $D$ :

$$\varphi_D = \Omega \circ \Lambda^2 \alpha_D : \Lambda^2 E_D \rightarrow \mathcal{O}_D.$$

Call this subscheme  $Z_\Omega$ . We denote by  $\overset{\circ}{Z}_\Omega$  the preimage of  $\overset{\circ}{Z} \subseteq Z$  under the projection  $\pi : Z_\Omega \rightarrow Z$ .

**Theorem 3.2.** *The restriction  $\overset{\circ}{\pi} : \overset{\circ}{Z}_\Omega \rightarrow \overset{\circ}{Z}$  is a closed embedding.*

*Remark 3.3.* The schemes we are dealing with are in fact of finite type; this means that we can apply Lemma 2.5 to prove the theorem. The morphism  $\overset{\circ}{\pi}$  is clearly proper, as a base change of a map between projective schemes. For injectivity, it is enough to note that there exists only one structure of symplectic sheaf on a given framed sheaf, in the sense of Remk. 2.9. It follows that we only need to prove the claim on tangent spaces: for any triple  $([q], A, \phi)$  the tangent map  $T_{([q], A, \phi)} \overset{\circ}{Z}_\Omega \rightarrow T_{([q], A)} \overset{\circ}{Z}$  is injective. This motivates the following infinitesimal study for the parameter spaces.

**3.3. Infinitesimal study: parameter spaces.** Let  $A$  be an artinian local  $\mathbb{K}$ -algebra. We define the sheaves on  $X_A = X \times \text{Spec}(A)$

$$\mathcal{H} := H \otimes \mathcal{O}_A = V \otimes \mathcal{O}_{X_A}(-m);$$

$$\mathcal{D} = \mathcal{O}_D \otimes \mathcal{O}_A \otimes W.$$

The aim of the present subsection is to compute the tangent spaces of the relative versions  $Z^A$  and  $Z_\Omega^A$  of the previously defined parameter spaces. In other words, we have

$$Z^A \subseteq \text{Quot}_{X_A}(\mathcal{H}, P) \times \mathbb{P}(\text{Hom}(\mathcal{H}, \mathcal{D})^\vee)$$

defined as the closed subscheme representing the functor assigning to an  $A$ -scheme  $T$  the set

$$\{q_T : V \otimes \mathcal{O}_{X_T}(-m) \twoheadrightarrow \tilde{\mathcal{E}}, \alpha_T : \tilde{\mathcal{E}} \rightarrow \mathcal{D}_T \mid \tilde{\mathcal{E}} \text{ } T\text{-flat}, P_{\tilde{\mathcal{E}}} = P\},$$

and

$$Z_\Omega^A \subseteq Z^A \times \mathbb{P}(\text{Hom}(\Lambda^2 \mathcal{H}, \mathcal{O}_{X_A})^\vee)$$

representing the functor assigning to an  $A$ -scheme  $T$  the set

$$\{(q_T, \alpha_T) \in Z^A(T), \varphi_T : \Lambda^2 \tilde{\mathcal{E}} \rightarrow \mathcal{O}_{X_T} \mid \varphi_T|_{D \times T} = \Omega \circ \Lambda^2 \alpha_T|_{D \times T}\}$$

The computation of the tangent spaces to  $Z^A$  was already performed in [HL], and goes as follows.

Let  $q : \mathcal{H} \rightarrow \mathcal{E}$  be a quotient with  $P_{\mathcal{E}} = P$ . Let  $\ker(q) := \mathcal{K} \xhookrightarrow{\iota} \mathcal{H}$ . The tangent space to  $\text{Quot}_{X_A}(\mathcal{H}, P)$  at the point  $[q : \mathcal{H} \rightarrow \mathcal{E}]$  is naturally isomorphic to the vector space  $\text{Hom}_{X_A}(\mathcal{K}, \mathcal{E})$ . Indeed, writing  $S = \text{Spec}(A[\varepsilon])$ , and using the universal property of  $\text{Quot}$ , the tangent space may be indeed identified with the set of equivalence classes of quotients  $\tilde{q} : \mathcal{H}_S \rightarrow \tilde{\mathcal{E}}$  on  $X_S$  (where  $\mathcal{H}_S = \mathcal{H} \otimes \mathbb{C}[\varepsilon] = H \otimes A[\varepsilon]$ , and  $\tilde{\mathcal{E}}$  an  $S$ -flat sheaf on  $X_S$ ) reducing to  $q \bmod \varepsilon$ . Write  $q_S : \mathcal{H}_S \rightarrow \mathcal{E}_S$  for the pullback of  $q$  to  $X_S$ . For a given  $\tilde{q}$  with  $\ker(\tilde{q}) = \tilde{\mathcal{K}}$ ,



the map  $q_S|_{\tilde{\mathcal{K}}}$  takes values in  $\varepsilon \cdot \mathcal{E}_S$  and factors through  $\mathcal{K}$ , defining a morphism  $\mathcal{K} \rightarrow \varepsilon \cdot \mathcal{E}_S \cong \mathcal{E}$ . Viceversa, a given map  $\gamma : \mathcal{K} \rightarrow \mathcal{E}$  defines

$$\tilde{\mathcal{K}} \subseteq \mathcal{H}_S, \tilde{\mathcal{K}} = \rho^{-1}(\mathcal{K}) \cap \ker(q_S + \gamma \circ \rho),$$

where  $\rho : \mathcal{H}_S \rightarrow \mathcal{H}$  is the natural projection induced by  $A[\varepsilon] \rightarrow A$ . The tangent space to  $\mathbb{P}(\text{Hom}(\mathcal{H}, \mathcal{D})^\vee)$  at a point  $[\mathcal{A}]$  and the tangent space to  $\mathbb{P}(\text{Hom}(\Lambda^2 \mathcal{H}, \mathcal{O}_A)^\vee)$  at a point  $[\Phi]$  are identified respectively with the quotients

$$\text{Hom}(\mathcal{H}, \mathcal{D})/A \cdot \mathcal{A}; \quad \text{Hom}(\Lambda^2 \mathcal{H}, \mathcal{O}_A)/A \cdot \Phi.$$

To see why this is the case, we apply again universal properties. An element of

$$\mathbb{P}(\text{Hom}(\mathcal{H}, \mathcal{D})^\vee)(S \rightarrow \text{Spec}(A)), \text{Spec}(\mathbb{K}) \mapsto [\mathcal{A}] \in \mathbb{P}(\text{Hom}(\mathcal{H}, \mathcal{D})^\vee)$$

is the same thing as a morphism  $\tilde{\mathcal{A}} : \mathcal{H}_S \rightarrow \mathcal{D}_S$  (up to units in  $A[\varepsilon]$ ), reducing to  $\mathcal{A} : \mathcal{H} \rightarrow \mathcal{D} \bmod \varepsilon$  (up to units in  $A$ ). Such a morphism may be represented as

$$h + \varepsilon h' \mapsto \mathcal{A}(h) + \varepsilon(\mathcal{A}(h') + \mathcal{B}(h))$$

with  $\mathcal{B} : \mathcal{H} \rightarrow \mathcal{D}$  an  $\mathcal{O}_{X_A}$ -linear map. The units in  $A[\varepsilon]$  preserving this map modulo  $\varepsilon$  are exactly those of the form  $1 + \lambda \varepsilon$  with  $\lambda \in A$ , and they send  $\mathcal{B}$  to  $\mathcal{B} + (\lambda - 1)\mathcal{A}$ . This proves the claim.

We proceed in an analogous way in the other case. An element of

$$\mathbb{P}(\text{Hom}(\Lambda^2 \mathcal{H}, \mathcal{O}_A)^\vee)(S \rightarrow \text{Spec}(A)), \text{Spec}(\mathbb{K}) \mapsto [\Phi] \in \mathbb{P}(\text{Hom}(\Lambda^2 \mathcal{H}, \mathcal{O}_A)^\vee)$$

is the same thing as a morphism  $\tilde{\Phi} : \Lambda^2 \mathcal{H}_S \rightarrow \mathcal{O}_S$  (up to units in  $A[\varepsilon]$ ), reducing to  $\Phi$  modulo  $\varepsilon$  (up to units in  $A$ ). Again, one can choose a representative

$$(h + \varepsilon h') \otimes (g + \varepsilon g') = \Phi(h \otimes g) + \varepsilon(\Phi(h' \otimes g) + \Phi(h \otimes g') + \psi(h \otimes g))$$

with  $\psi : \Lambda^2 \mathcal{H} \rightarrow \mathcal{O}_{X_A}$ , and two different maps  $\psi$  will induce the same morphism up to units if and only if they differ by an  $A$ -multiple of  $\Phi$ .

Let  $(q, \alpha) \in Z^A$  with  $\alpha \circ q = \mathcal{A}$ . The pairs

$$(\gamma, \mathcal{B}) \in \text{Hom}(\mathcal{K}, \mathcal{E}) \oplus \text{Hom}(\mathcal{H}, \mathcal{D})/A \cdot \mathcal{A} = T_{(q, \mathcal{A})}(\text{Quot}(\mathcal{H}, P) \times \mathbb{P}(\text{Hom}(\mathcal{H}, \mathcal{D})^\vee))$$

belonging to  $T_{(q, \alpha)}Z^A$  are characterized by the equation

$$\mathcal{B} \circ \iota = \alpha \circ \gamma,$$

see [HL].

Now, let  $(\tilde{q}, \tilde{\mathcal{A}}, \tilde{\Phi}) \in Z_H^A(S)$ . This means that the morphism  $\tilde{\Phi} : \Lambda^2 \mathcal{H}_S \rightarrow \mathcal{O}_S$  descends to some  $\tilde{\varphi} : \Lambda^2 \mathcal{E}_S \rightarrow \mathcal{O}_S$  via  $\tilde{q}$ , i.e. its restriction to  $\tilde{\mathcal{K}} \otimes \mathcal{H}_S$  vanishes. In local sections, an element  $h + \varepsilon h' \in \mathcal{H}_S$  belongs to  $\tilde{\mathcal{K}}$  if and only if  $q(h) = 0$  and  $\gamma(h) = -q(h')$ . We obtain:

$$\tilde{\Phi}((h + \varepsilon h') \otimes (g + \varepsilon g')) = \varepsilon(\Phi(h' \otimes g) + \psi(h \otimes g)) = \varepsilon(-\varphi(\gamma(h) \otimes q(g)) + \psi(h \otimes g)).$$

This quantity vanishes if and only if the equation

$$\psi(\iota \otimes 1) = \varphi(\gamma \otimes q)$$

holds.

The triple is required to satisfy another condition, namely the compatibility on the divisor  $D_S$ :

$$(\tilde{\varphi})|_{D_S} = \Omega \circ (\tilde{\mathcal{A}}^{\otimes 2})|_{D_S}.$$

We make an abuse of notation by writing  $h + \varepsilon h'$  for sections of  $\mathcal{H}_S|_{D_S} \cong V \otimes \mathcal{O}_{D_S}(-m)$ ; we get

$$\tilde{\Phi}((h + \varepsilon h') \otimes (g + \varepsilon g')) = \Omega((\mathcal{A}(h) + \varepsilon(\mathcal{A}(h') + \mathcal{B}(h))) \otimes (\mathcal{A}(g) + \varepsilon(\mathcal{A}(g') + \mathcal{B}(g))))$$

$$\begin{aligned} \Phi(h \otimes g) + \varepsilon(\Phi(h' \otimes g) + \Phi(h \otimes g') + \psi(h \otimes g)) &= \Omega(\mathcal{A}(h) \otimes \mathcal{A}(g)) + \\ + \varepsilon(\Omega(\mathcal{A}(h) \otimes \mathcal{A}(g')) + \Omega(\mathcal{A}(h) \otimes \mathcal{B}(g)) + \Omega(\mathcal{A}(h') \otimes \mathcal{A}(g)) + \Omega(\mathcal{B}(h) \otimes \mathcal{A}(g))) \end{aligned}$$

Since

$$(\varphi)|_{D_S} = \Omega \circ (\mathcal{A}^{\otimes 2})|_{D_S}, \quad \alpha \circ q = \mathcal{A}$$

holds by hypothesis, we can simplify:

$$\psi(h \otimes g) = \Omega(\alpha \circ q(h) \otimes \mathcal{B}(g)) + \Omega(\mathcal{B}(h) \otimes \alpha \circ q(g)).$$

We have obtained the description of the tangent space we needed.

**Proposition 3.4.** *There are isomorphisms:*

$$\begin{aligned} T_{(q, \mathcal{A})} Z^A &\cong \{(\gamma, \mathcal{B}) \in \text{Hom}(\mathcal{K}, \mathcal{E}) \oplus \text{Hom}(\mathcal{H}, \mathcal{D}) / A \cdot \mathcal{A} \mid \mathcal{B} \circ \iota = \alpha \circ \gamma\}; \\ T_{(q, \mathcal{A}, \Phi)} Z_\Omega^A &\cong \{((\gamma, \mathcal{B}), \psi) \in T_{(q, \mathcal{A})} Z^A \oplus \text{Hom}(\Lambda^2 \mathcal{H}, \mathcal{O}_A) / A \cdot \Phi \mid \psi(\iota \otimes 1) = \varphi(\gamma \otimes q), \\ &\quad \psi|_{D_A} = \Omega(\mathcal{A} \otimes \mathcal{B} + \mathcal{B} \otimes \mathcal{A})\} \end{aligned}$$

**3.4. Moduli spaces.** We can now apply the results in the previous subsection to prove Thm. 3.2.

*Proof.* For a quotient  $q : H \rightarrow E$ , we denote by  $\iota : K \rightarrow H$  its kernel. We know that the tangent space  $T_{([q], A)} Z$ , which is naturally a subspace of  $\text{Hom}(K, E) \oplus (\text{Hom}(H, D_W)/\mathbb{C}A)$ , can be described as the subset of pairs  $(\gamma, [B])$  satisfying the equation

$$\bar{A} \circ \gamma = B|_K.$$

The space  $T_{([q], A, \phi)} \subseteq T_{([q], A)} Z \oplus (\text{Hom}(\Lambda^2 H, \mathcal{O}_X)/\mathbb{K}\phi)$  can be instead identified with the subspace of triples  $(\gamma, [B], [\psi])$  defined by the equations

$$\psi(\iota \otimes 1_H) = \varphi(\gamma \otimes q); \quad \psi_D = \Omega(A \otimes B + B \otimes A).$$

The differential map  $T_{([q], A, \phi)} \overset{\circ}{Z}_\Omega \rightarrow T_{([q], A)} \overset{\circ}{Z}$  is just the projection

$$(\gamma, [B], [\psi]) \mapsto (\gamma, [B]).$$

Now, suppose an element of type  $(0, \lambda\mathcal{A}, [\psi])$  belongs to  $T_{([q], A, \phi)} \overset{\circ}{Z}_\Omega$ . This means

$$\psi(\iota \otimes 1_\Omega) = 0,$$

so that  $\psi$  descends to some  $\bar{\psi} \in \text{Hom}(\Lambda^2 E, \mathcal{O}_X)$ . Furthermore, we get

$$\psi_D = \lambda^2 \Omega(\mathcal{A}^{\otimes 2} + \mathcal{A}^{\otimes 2} \circ i) = 0 \implies \psi_D \in \text{Hom}(\Lambda^2 E, \mathcal{O}_X(-D)) = 0.$$

We have proved injectivity for the tangent map, and this finishes the proof thanks to Remk. 3.3.  $\square$

The natural action of the group  $SL(V)$  on the bundle  $H = V \otimes \mathcal{O}_X(-m)$  induces  $SL(V)$  actions on the schemes  $Z$  and  $Z_\Omega$ , and the map  $Z_\Omega \rightarrow Z$  is equivariant. In addition, the open subschemes  $\overset{\circ}{Z}$  and  $\overset{\circ}{Z}_\Omega$  are invariant. As already announced, the following theorem holds:

**Theorem 3.5.** *The  $SL(V)$ -scheme  $\overset{\circ}{Z}$  admits a geometric quotient  $\overset{\circ}{Z}/SL(V)$ ; this scheme is a fine moduli space for the functor  $\mathfrak{M}_X^D(r, n)$ , and will be denoted  $\mathcal{M}_X^D(r, n)$ .*

**Definition 3.6.** We define the scheme  $\mathcal{M}_{X, \Omega}^D(r, n)$  to be the closed subscheme

$$\overset{\circ}{Z}_\Omega/SL(V) \subseteq \mathcal{M}_X^D(r, n).$$

**Theorem 3.7.** The scheme  $\mathcal{M}_{X, \Omega}^D(r, n)$  is a fine moduli space for the functor  $\mathfrak{M}_X^D(r, n)$ .

*Proof.* Let  $S$  be any scheme of finite type and let  $(\mathcal{E}_S, a_S, \varphi_S)$  be an  $S$ -family of framed symplectic sheaves. Consider the sheaf  $\mathcal{V}_S := p_{S*}(\mathcal{E}_S \otimes p_X^* \mathcal{O}_X(m))$ . Since for any  $s \in S(\mathbb{K})$  we have

$$H^i(\mathcal{E}_S \otimes p_X^* \mathcal{O}_X(m) |_{\{s\} \times X}) = 0 \quad \forall i > 0$$

by Lemma 3.1,  $\mathcal{V}_S$  is locally of rank  $P(m)$ . Furthermore, the natural map

$$q_S : p_S^* \mathcal{V}_S \rightarrow \mathcal{E}_S \otimes p_X^* \mathcal{O}_X(m)$$

is surjective, as it restricts to

$$H^0(\mathcal{E}_S \otimes p_X^* \mathcal{O}_X(m) |_{\{s\} \times X}) \otimes \mathcal{O}_X \rightarrow \mathcal{E}_S \otimes p_X^* \mathcal{O}_X(m) |_{\{s\} \times X}$$

on fibres, and we may apply again Lemma 3.1. From  $\varphi_S$  we obtain a map

$$\mathcal{V}_S^{\otimes 2} \rightarrow p_{S*}(\mathcal{O}_{X_S} \otimes p_X^* \mathcal{O}_X(2m)) = \mathcal{O}_S \otimes H^0(X, \mathcal{O}_X(2m)).$$

Indeed, since the higher  $p_S$ -pushforwards of  $\mathcal{E}_S \otimes p_X^* \mathcal{O}_X(m)$  vanish, the formula

$$Rp_{T*}(\mathcal{E}_S \otimes p_X^* \mathcal{O}_X(m) \overset{L}{\otimes} \mathcal{E}_S \otimes p_X^* \mathcal{O}_X(m)) \cong Rp_{S*}(\mathcal{E}_S \otimes p_X^* \mathcal{O}_X(m)) \overset{L}{\otimes} Rp_{S*}(\mathcal{E}_S \otimes p_X^* \mathcal{O}_X(m)) \cong \mathcal{V}_S^{\otimes 2}$$

holds, and induces the desired map. It still is skew-symmetric, and thus yields

$$p_{S*}(\varphi_S \otimes p_X^* \mathcal{O}_X(2m)) : \Lambda^2 \mathcal{V}_S \rightarrow \mathcal{O}_S \otimes H^0(X, \mathcal{O}_X(2m)).$$

We define the morphism  $\phi_S$  to be the composition

$$\phi_S : \Lambda^2 p_X^* \mathcal{V}_S \rightarrow \mathcal{O}_{X_S} \otimes H^0(X, \mathcal{O}_X(2m)) \rightarrow \mathcal{O}_{X_S} \otimes p_X^* \mathcal{O}_X(2m).$$

Also  $a_S$  similarly induces a map

$$\mathcal{V}_S \rightarrow \mathcal{O}_S \otimes H^0(X, \mathcal{O}_D(m) \otimes W)$$

and we define as above

$$A_S : p_X^* \mathcal{V}_S \rightarrow \mathcal{O}_{D_S} \otimes p_X^* \mathcal{O}_X(m) \otimes W.$$

By construction, the diagrams

$$\begin{array}{ccc} \Lambda^2 p_X^* \mathcal{V}_S & \xrightarrow{\phi_S} & \mathcal{O}_{X_S} \otimes p_X^* \mathcal{O}_X(m) \\ \Lambda^2 q_S \downarrow & \nearrow \varphi_S(2m) & \\ \Lambda^2 \mathcal{E}_S \otimes p_X^* \mathcal{O}_X(2m) & & \end{array} \quad \begin{array}{ccc} p_X^* \mathcal{V}_S & \xrightarrow{A_S} & \mathcal{O}_{D_S} \otimes p_X^* \mathcal{O}_X(m) \otimes W \\ q_S \downarrow & \nearrow a_S(m) & \\ \mathcal{E}_S \otimes p_X^* \mathcal{O}_X(m) & & \end{array}$$

are commutative.

Define an open covering  $S = \bigcup S_i$  such that  $\mathcal{V}_S$  trivializes over each  $S_i$ , and fix isomorphisms  $\mathcal{O}_{S_i} \otimes V \cong \mathcal{V}_{S_i}$ , where  $V$  is a vector space of a dimension  $P(m)$ ; the trivializations differ on the overlaps  $S_{ij}$  by a map  $S_{ij} \rightarrow GL(V)$ . Restricting the maps  $q_S$ ,  $\phi_S$  and  $A_S$  to  $S_i \times X$  and twisting by  $p_X^* \mathcal{O}_X(-m)$  we obtain maps  $S_i \rightarrow \mathring{Z}_\Omega$ , which glue to a map  $S \rightarrow \mathring{Z}_\Omega / SL(V) = \mathcal{M}_{X,\Omega}^D(r, n)$ . We note that acting via  $SL(V)$  or  $GL(V)$  does not make a real difference as the natural action by  $\mathbb{G}_m$  on the parameter spaces is trivial.

The resulting natural transformation

$$\mathfrak{M}_{X,\Omega}^D(r, n) \rightarrow \mathcal{M}_{X,\Omega}^D(r, n)$$

makes  $\mathcal{M}_{X,\Omega}^D(r, n)$  into a coarse moduli space for framed symplectic sheaves; indeed, since  $\mathring{Z}_\Omega$  parameterizes a tautological family of framed symplectic sheaves, for any scheme  $N$  and for any natural transformation  $\mathfrak{M}_{X,\Omega}^D(r, n) \rightarrow Hom(\_, N)$  we obtain a map  $\mathring{Z}_\Omega \rightarrow N$ . This map has to be  $SL(V)$  invariant as two points of  $\mathring{Z}_\Omega$  that lie in the same orbit define isomorphic framed sheaves. The fact that the moduli space is indeed fine can be proved by noting that framed symplectic sheaves are rigid, i.e. by applying Remk. 2.9 and proceeding as in [HL, proof of Main Theorem].  $\square$

*Remark 3.8.* The moduli spaces  $\mathcal{M}_X^D(r, n)$  and  $\mathcal{M}_{X,\Omega}^D(r, n)$  both contain open subschemes of isomorphism classes of locally free sheaves. These are fine moduli spaces for framed  $SL_r$  and  $SP_r$  principal bundles, and will be respectively denoted  $\mathcal{M}_X^{D,reg}(r, n)$  and  $\mathcal{M}_{X,\Omega}^{D,reg}(r, n)$  in the sequel.

We can now apply the results of the previous subsection to describe the tangent spaces to  $\mathcal{M}_{X,\Omega}^D(r, n)$ .

**3.5. Infinitesimal study: moduli spaces.** The aim of this subsection is to prove the following theorem.

**Theorem 3.9.** *Let  $\xi = [E, a, \varphi] \in \mathcal{M}_{X,\Omega}^D(r, n)$ . The tangent space  $T_\xi \mathcal{M}_{X,\Omega}^D(r, n)$  is naturally isomorphic to the kernel of a canonically defined linear map*

$$p_\varphi : \text{Ext}_{\mathcal{O}_X}^1(E, E(-D)) \rightarrow \text{Ext}_{\mathcal{O}_X}^1(\Lambda^2 E, \mathcal{O}_X(-D)).$$

Let us return to the setting and notation of subsection 3.3. Consider the natural actions of  $\text{Aut}(\mathcal{H})$  on  $\overset{\circ}{Z}^A$  and  $\overset{\circ}{Z}_\Omega^A$ . To obtain the tangent spaces to  $\overset{\circ}{Z}^A/\text{Aut}(\mathcal{H})$  at a point  $[(q, \mathcal{A})]$  and to  $\overset{\circ}{Z}_\Omega^A/\text{Aut}(\mathcal{H})$  at  $[(q, \mathcal{A}, \Phi)]$  one has to mod out the image of the induced tangent orbit maps

$$\text{End}(\mathcal{H}) \rightarrow T_{(q, \mathcal{A})} \overset{\circ}{Z}^A, \quad \text{End}(\mathcal{H}) \rightarrow T_{(q, \mathcal{A}, \Phi)} \overset{\circ}{Z}_\Omega^A.$$

These tangent maps factor through

$$\text{End}(\mathcal{H}) \rightarrow \text{Hom}(\mathcal{H}, \mathcal{E}), \quad x \mapsto q \circ x$$

and are described by

$$\text{Hom}(\mathcal{H}, \mathcal{E}) \ni \lambda \mapsto (\lambda \circ \iota, \alpha \circ \lambda) \in T_{(q, \mathcal{A})} \overset{\circ}{Z}^A;$$

$$\text{Hom}(\mathcal{H}, \mathcal{E}) \ni \lambda \mapsto (\lambda \circ \iota, \alpha \circ \lambda, \varphi(\lambda \otimes q + q \otimes \lambda)) \in T_{(q, \mathcal{A})} \overset{\circ}{Z}_\Omega^A.$$

In the first case, the quotient can be shown to be isomorphic to the hyperext group

$$\mathbb{E}xt^1(\mathcal{E}, \mathcal{E} \rightarrow \mathcal{D}).$$

To prove this, one interprets a pair  $(\gamma, \mathcal{B}) \in T_{(q, \mathcal{A})} \overset{\circ}{Z}^A$  as a morphism of complexes

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{\iota} & \mathcal{H} \\ \gamma \downarrow & & \downarrow \mathcal{B} \\ \mathcal{E} & \xrightarrow{\mathcal{A}} & \mathcal{D} \end{array}$$

and it is immediate to see that the subspace of nullhomotopic morphisms coincides with the image of  $\text{Hom}(\mathcal{H}, \mathcal{E}) \rightarrow T_{(q, \mathcal{A})} \overset{\circ}{Z}^A$ . We obtain a chain of natural isomorphisms

$$T_{(q, \mathcal{A})} \overset{\circ}{Z}^A / \text{Hom}(\mathcal{H}, \mathcal{E}) \cong \text{Hom}_K(\mathcal{K} \rightarrow \mathcal{H}, \mathcal{E} \rightarrow \mathcal{D}) \cong \mathbb{E}xt^1(\mathcal{E}, \mathcal{E} \rightarrow \mathcal{D}),$$

see [HL]. Fix a symplectic form  $\varphi : \Lambda^2 \mathcal{E} \rightarrow \mathcal{O}_{X_A}$ . We define a map

$$p_\varphi : \text{Hom}_K(\mathcal{K} \rightarrow \mathcal{H}, \mathcal{E} \rightarrow \mathcal{D}) \rightarrow \text{Hom}_K(\mathcal{K} \wedge \mathcal{H} \rightarrow \Lambda^2 \mathcal{H}, \mathcal{O}_{X_A} \rightarrow \mathcal{O}_{D_A})$$

by assigning to a morphism of complexes  $(\gamma, \mathcal{B})$  the morphism

$$\varphi(\gamma \otimes q) : \mathcal{K} \wedge \mathcal{H} \rightarrow \mathcal{O}_{X_A}, \quad \Omega(\alpha q \otimes \mathcal{B} + \mathcal{B} \otimes \alpha q) : \Lambda^2 \mathcal{H} \rightarrow \mathcal{O}_{D_A}.$$

The map  $\varphi(\gamma \otimes q)$  is naturally defined on  $\mathcal{K} \otimes \mathcal{H}$  but, due to the skew-symmetry of  $\varphi$ , it vanishes on the subsheaf  $S^2 \mathcal{H} \cap (\mathcal{K} \otimes \mathcal{H})$ ; we use the same notation for the naturally induced map on the quotient  $\mathcal{K} \wedge \mathcal{H}$ , see Lemma 2.2.  $p_\varphi$  is well defined since if  $(\gamma, \mathcal{B})$  is homotopic to 0 and  $\lambda : \mathcal{H} \rightarrow \mathcal{E}$  is an homotopy, then  $\varphi(\lambda \otimes q + q \otimes \lambda)$  will be an homotopy for  $p_\varphi(\gamma, \mathcal{B})$ .

By definition,  $p_\varphi(\gamma, \mathcal{B}) = 0$  if and only if there exists a morphism  $\psi : \Lambda^2 \mathcal{H} \rightarrow \mathcal{O}_{X_A}$  such that  $\psi(\iota \otimes 1) = \varphi(\gamma \otimes q)$  and  $\psi|_{D_A} = \Omega(\alpha q \otimes \mathcal{B} + \mathcal{B} \otimes \alpha q)$ .

*Remark 3.10.*  $\mathbb{E}xt^1(\mathcal{E}, \mathcal{E} \rightarrow \mathcal{O}_D \otimes W)$  is in fact isomorphic to  $\mathbb{E}xt^1(\mathcal{E}, \mathcal{E}(-D_A))$  as  $\mathcal{E}$  is locally free on  $D_A$  by hypothesis.

*Remark 3.11.* The natural map

$$\mathrm{Hom}_K(\mathcal{K} \wedge \mathcal{H} \rightarrow \Lambda^2 \mathcal{H}, \mathcal{O}_{X_A} \rightarrow \mathcal{O}_{D_A}) \rightarrow \mathbb{H}om_{D^b}(\mathcal{K} \wedge \mathcal{H} \rightarrow \Lambda^2 \mathcal{H}, \mathcal{O}_{X_A} \rightarrow \mathcal{O}_{D_A})$$

is an isomorphism.

*Proof.* We prove surjectivity first. We represent elements of the target group as roofs in the derived category, i.e. as pairs given by a quasi-isomorphism  $M^\bullet \rightarrow (\mathcal{K} \wedge \mathcal{H} \rightarrow \Lambda^2 \mathcal{H})$  and a morphism  $M^\bullet \rightarrow (\mathcal{O}_{X_A} \rightarrow \mathcal{O}_{D_A})$ , where  $M^\bullet$  is a complex concentrated in degrees zero and one. We get a morphism of short exact sequences

$$\begin{array}{ccccc} M^0 & \longrightarrow & M^1 & \longrightarrow & \Lambda^2 \mathcal{E} \\ \downarrow & & \downarrow & & \parallel \\ \mathcal{K} \wedge \mathcal{H} & \longrightarrow & \Lambda^2 \mathcal{H} & \longrightarrow & \Lambda^2 \mathcal{E} \end{array}$$

Apply  $\mathrm{Hom}(\_, \mathcal{O}_{X_A})$  and use

$$\mathbb{E}xt^1(\Lambda^2 \mathcal{H}, \mathcal{O}_{X_A}) \cong \Lambda^2 V \otimes H^1(X_A, \mathcal{O}_{X_A}(2m)) = 0;$$

get

$$\begin{array}{ccccccc} \mathrm{Hom}(\Lambda^2 \mathcal{E}, \mathcal{O}_{X_A}) & \longrightarrow & \mathrm{Hom}(\Lambda^2 \mathcal{H}, \mathcal{O}_{X_A}) & \longrightarrow & \mathrm{Hom}(\mathcal{K} \wedge \mathcal{H}, \mathcal{O}_{X_A}) & \twoheadrightarrow & \mathbb{E}xt^1(\Lambda^2 \mathcal{E}, \mathcal{O}_{X_A}) \\ \parallel & & \downarrow & & \downarrow & & \parallel \\ \mathrm{Hom}(\Lambda^2 \mathcal{E}, \mathcal{O}_{X_A}) & \longrightarrow & \mathrm{Hom}(M^1, \mathcal{O}_{X_A}) & \longrightarrow & \mathrm{Hom}(M^0, \mathcal{O}_{X_A}) & \longrightarrow & \mathbb{E}xt^1(\Lambda^2 \mathcal{E}, \mathcal{O}_{X_A}) \end{array}$$

We deduce that the natural map

$$\mathrm{Hom}(\mathcal{K} \wedge \mathcal{H}, \mathcal{O}_{X_A}) \rightarrow \mathrm{Hom}(M^0, \mathcal{O}_{X_A}) / \mathrm{Hom}(M^1, \mathcal{O}_{X_A})$$

is surjective; in other words we can suppose that the morphism  $M^0 \rightarrow \mathcal{O}_{X_A}$  comes from  $\mathrm{Hom}(\mathcal{K} \wedge \mathcal{H}, \mathcal{O}_{X_A})$  up to homotopy. To show that also  $M^1 \rightarrow \mathcal{O}_{D_A}$  comes from a map  $\Lambda^2 \mathcal{H} \rightarrow \mathcal{O}_{D_A}$ , we may proceed as above: apply  $\mathrm{Hom}(\_, \mathcal{O}_{D_A})$  and use

$$\mathbb{E}xt^1(\Lambda^2 \mathcal{H}, \mathcal{O}_{D_A}) \cong \Lambda^2 V \otimes H^1(X_A, \mathcal{O}_{D_A}(2m)) = 0.$$

This proves surjectivity.

To prove injectivity, let  $f \in \mathrm{Hom}_K(\mathcal{K} \wedge \mathcal{H} \rightarrow \Lambda^2 \mathcal{H}, \mathcal{O}_{X_A} \rightarrow \mathcal{O}_{D_A})$  be such that there exists a quasi isomorphism  $M^\bullet \rightarrow (\mathcal{K} \wedge \mathcal{H} \rightarrow \Lambda^2 \mathcal{H})$  whose composition with  $c$  admits a homotopy  $h : M^1 \rightarrow \mathcal{O}_{X_A}$ . We only need to prove that  $h$  factors through another homotopy  $\Lambda^2 \mathcal{H} \rightarrow \mathcal{O}_{X_A}$ ; this is again achieved by an easy chasing of the diagram above.  $\square$

Since there is an obvious isomorphism

$$\mathbb{H}om_{D^b}(\mathcal{K} \wedge \mathcal{H} \rightarrow \Lambda^2 \mathcal{H}, \mathcal{O}_{X_A} \rightarrow \mathcal{O}_{D_A}) \cong Ext^1(\Lambda^2 \mathcal{E}, \mathcal{O}_{X_A}(-D_A)),$$

the previous discussion leads us to conclude that the tangent space  $T_\Omega^{1,A}$  to  $\mathring{Z}_\Omega^A/Aut(\mathcal{H})$  at  $[(q, \mathcal{A}, \Phi)]$  fits into an exact sequence of vector spaces

$$0 \rightarrow T_\Omega^{1,A} \rightarrow Ext^1(\mathcal{E}, \mathcal{E}(-D_A)) \rightarrow Ext^1(\Lambda^2 \mathcal{E}, \mathcal{O}_{X_A}(-D_A)).$$

*Remark 3.12.* Let  $A = \mathbb{K}$ . The map  $p_\varphi$  is in fact canonical (i.e., it only depends on the triple  $(E, \varphi, a)$ ) since it admits the following Yoneda-type description. Let  $\xi \in Ext^1(E, E(-D))$  be represented by an extension

$$0 \longrightarrow E(-D) \xrightarrow{\iota} F \xrightarrow{\pi} E \longrightarrow 0.$$

Apply  $E \otimes \_$  and pushout via  $\varphi(-D)$ :

$$\begin{array}{ccccccc} 0 & \longrightarrow & ker(1_E \otimes \iota) & \longrightarrow & E^{\otimes 2}(-D) & \xrightarrow{1_E \otimes \iota} & E \otimes F \longrightarrow E^{\otimes 2} \longrightarrow 0 \\ & & \downarrow & & \downarrow \varphi(-D) & & \downarrow & & \parallel \\ 0 & \longrightarrow & ker(\chi) & \longrightarrow & \mathcal{O}_X(-D) & \xrightarrow{\chi} & M \xrightarrow{p} E^{\otimes 2} \longrightarrow 0 \end{array}$$

Now, the module  $ker(1_E \otimes \iota)$  is a torsion sheaf; it is indeed an epimorphic image of the 0-dimensional sheaf  $\mathcal{T}or_1(E, E)$ . It follows that  $ker(\chi)$  is torsion as well: it is then forced to vanish. The above construction defines a linear map

$$\varphi(1_E \otimes \_) : Ext^1(E, E(-D)) \rightarrow Ext^1(E^{\otimes 2}, \mathcal{O}_X(-D)).$$

Similarly, define the “adjoint” map  $\varphi(\_ \otimes 1_E)$ . It is immediate to verify that the map

$$\varphi(\_ \otimes 1_E) + \varphi(1_E \otimes \_) : Ext^1(E, E(-D)) \rightarrow Ext^1(E^{\otimes 2}, \mathcal{O}_X(-D))$$

takes values in the subspace of skew extensions

$$Ext^1(\Lambda^2 E, \mathcal{O}_X(-D)) \subseteq Ext^1(E^{\otimes 2}, \mathcal{O}_X(-D)),$$

and a direct check shows that the equation

$$\varphi(\_ \otimes 1_E) + \varphi(1_E \otimes \_) = p_\varphi$$

holds.

**Corollary 3.13.** *Let  $(E, \alpha, \varphi)$  be a framed symplectic sheaf whose underlying framed sheaf  $(E, \alpha)$  corresponds to a smooth point  $[(E, \alpha)] \in \mathcal{M}_X^D(r, n)$ , and suppose that the map  $p_\varphi$  is an epimorphism. Then  $[(E, \alpha, \varphi)]$  is a smooth point of  $\mathcal{M}_{X, \Omega}^D(r, n)$ .*

*Proof.* Let  $A$  be an artinian local  $\mathbb{K}$ -algebra and let  $(\mathcal{E}, \mathcal{A}, \Phi) \in \mathfrak{M}_{X, \Omega}^D(A)$  be a framed symplectic sheaf over  $X_A$ . We proved that the space of its infinitesimal deformations can be

written as

$$T^1(\mathcal{E}, \mathcal{A}, \Phi)_A = \ker(\text{Ext}_{\mathcal{O}_{X_A}}^1(\mathcal{E}, \mathcal{E}(-D_A)) \rightarrow \text{Ext}_{\mathcal{O}_{X_A}}^1(\Lambda^2 \mathcal{E}, \mathcal{O}_{X_A}(-D_A))).$$

We want to give a sufficient condition for the smoothness at a closed point  $(E, \alpha, \varphi)$  by means of the  $T^1$ -lifting property. In our setting, this property may be expressed in the following way. Let  $A_n \cong \mathbb{C}[t]/t^{n+1}$ ,  $n \in \mathbb{N}$ . Let  $(\mathcal{E}_n, \mathcal{A}_n, \Phi_n) \in \mathfrak{M}_{X, \Omega}^D(A_n)$  and  $(\mathcal{E}_{n-1}, \mathcal{A}_{n-1}, \Phi_{n-1}) \in \mathfrak{M}_{X, \Omega}^D(A_{n-1})$  be its pullback via the natural map  $A_n \twoheadrightarrow A_{n-1}$ . We get a map

$$T^1(\mathcal{E}_n, \mathcal{A}_n, \Phi_n)_{A_n} \rightarrow T^1(\mathcal{E}_{n-1}, \mathcal{A}_{n-1}, \Phi_{n-1})_{A_{n-1}}$$

and the underlying closed point  $[(E, \alpha, \varphi)]$ , where  $(E, \alpha, \varphi) = (\mathcal{E}_n, \mathcal{A}_n, \Phi_n) \bmod(t)$ , turns out to be a smooth point of  $\mathcal{M}_{X, \Omega}^D(r, n)$  if and only if the above map is surjective for any  $n$ . From the exact sequence of  $\mathcal{O}_{X_n} := \mathcal{O}_{X_{A_n}}$  modules

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{X_n} \rightarrow \mathcal{O}_{X_{n-1}} \rightarrow 0$$

we get exact sequences

$$0 \rightarrow E \rightarrow \mathcal{E}_n \rightarrow \mathcal{E}_{n-1} \rightarrow 0$$

and

$$0 \rightarrow \Lambda^2 E \rightarrow \Lambda^2 \mathcal{E}_n \rightarrow \Lambda^2 \mathcal{E}_{n-1} \rightarrow 0.$$

Making extensive use of Lemma 2.4, we obtain a commutative diagram with exact rows and columns

$$\begin{array}{ccccccc} 0 & \longrightarrow & T^1 & \longrightarrow & \text{Ext}_{\mathcal{O}_X}^1(E, E(-D)) & \xrightarrow{h} & \text{Ext}_{\mathcal{O}_X}^1(\Lambda^2 E, \mathcal{O}_X(-D)) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & T^{1,n} & \longrightarrow & \text{Ext}_{\mathcal{O}_{X_n}}^1(\mathcal{E}_n, \mathcal{E}_n(-D_n)) & \xrightarrow{p_{n,\varphi}} & \text{Ext}_{\mathcal{O}_{X_n}}^1(\Lambda^2 \mathcal{E}_n, \mathcal{O}_{X_n}(-D_n)) \\ & & \downarrow a & & \downarrow b & & \downarrow c \\ 0 & \longrightarrow & T^{1,n-1} & \longrightarrow & \text{Ext}_{\mathcal{O}_{X_{n-1}}}^1(\mathcal{E}_{n-1}, \mathcal{E}_{n-1}(-D_{n-1}) \otimes W) & \longrightarrow & \text{Ext}_{\mathcal{O}_{X_{n-1}}}^1(\Lambda^2 \mathcal{E}_{n-1}, \mathcal{O}_{X_{n-1}}(-D_{n-1})) \end{array}$$

Call  $\tilde{c}$  the restriction of  $c$  to the image of  $p_{n,\varphi}$  in the diagram; by applying the snake lemma to the second and third row, we get an exact sequence of vector spaces

$$\ker(a) \rightarrow \ker(b) \rightarrow \ker(\tilde{c}) \rightarrow \text{coker}(a) \rightarrow \text{coker}(b) \rightarrow \text{coker}(\tilde{c}).$$

We know by hypothesis that  $\text{coker}(b) = 0$ . A sufficient condition to get or claim  $\text{coker}(a) = 0$  is  $\ker(b) \rightarrow \ker(c)$  be surjective ( $\implies \ker(c) = \ker(\tilde{c})$ ). This condition is clearly satisfied if  $h$  is surjective.  $\square$

We conclude the section with a direct application to the case of bundles.



**Corollary 3.14.** *If  $(E, \alpha, \varphi)$  is a framed symplectic bundle, the corresponding point  $[(E, \alpha, \varphi)] \in \mathcal{M}_{X, \Omega}^{D, reg}(r, n)$  is smooth if  $[(E, \alpha)] \in \mathcal{M}_X^{D, reg}(r, n)$  is.*

*Proof.* Let  $(E, \alpha, \varphi)$  be a symplectic bundle. Consider the following map:

$$\mathcal{H}om(E, E(-D)) \rightarrow \mathcal{H}om(\Lambda^2 E, \mathcal{O}(-D)) \subseteq \mathcal{H}om(E, E^\vee(-D))$$

defined on sections by

$$f \mapsto \varphi(-D) \circ f + (f(-D))^\vee \varphi,$$

where  $\varphi$  is interpreted as an isomorphism  $E \rightarrow E^\vee$ . The kernel of this map is identified with the bundle  $Ad^\varphi(E)(-D)$ , i.e. the twisted adjoint bundle associated with the principal  $SP$ -bundle defined by  $(E, \varphi)$ . The map defined above is easily proved to be surjective since  $\varphi$  is an isomorphism. We obtain a short exact sequence of bundles which is in fact split-exact, as the map

$$\mathcal{H}om(\Lambda^2 E, \mathcal{O}(-D)) \rightarrow \mathcal{H}om(E, E(-D)), \psi \mapsto \frac{1}{2}\psi(\varphi(-D))^{-1}$$

gives a splitting. In particular, the  $H^1$ -factors of the corresponding long exact sequence in cohomology define an exact sequence

$$0 \rightarrow H^1(Ad^\varphi(E)(-D)) \rightarrow Ext^1(E, E(-D)) \rightarrow Ext^1(\Lambda^2 E, \mathcal{O}(-D)) \rightarrow 0$$

whose second map is just the map  $p_\varphi$ . The surjectivity of the latter provides the result. We remark that the result does not require the natural obstruction space  $H^2(X, Ad^\varphi(-D))$ , from the theory of principal bundles, to vanish.  $\square$

#### 4. MODULI SPACES OF FRAMED SYMPLECTIC SHEAVES ON $\mathbb{P}_{\mathbb{C}}^2$

We fix now  $\mathbb{K} = \mathbb{C}$ . The moduli space of framed sheaves on  $X = \mathbb{P}^2$  (where the framing divisor is a line) has a description in terms of the so called *ADHM data*, which means that it can be realized as a quotient of certain spaces of matrices. In the present section we prove that an analogous result holds in the symplectic case. In the next section, we will apply this result for proving that our moduli space is irreducible.

**4.1. ADHM data and monads.** Let  $r, n$  be positive integers, and let  $W, V$  be complex vector spaces with  $\dim(W) = r, \dim(V) = n$ .

**Definition 4.1.** The *variety of ADHM data* of type  $(r, n)$  is the closed subvariety of the affine space

$$End(V)^{\oplus 2} \oplus Hom(W, V) \oplus Hom(V, W)$$

defined by

$$\mathbb{M}(r, n) = \{(A, B, I, J) \mid [A, B] + IJ = 0\}.$$

The equation  $[A, B] + IJ = 0$  will be called *ADHM equation*.

The group  $GL(V)$  acts on  $\mathbb{M}(r, n)$  naturally:

$$g \cdot (A, B, I, J) = (gAg^{-1}, gBg^{-1}, gI, Jg^{-1}).$$

**Definition 4.2.** We call an ADHM datum  $(A, B, I, J)$

- *stable* if there exists no proper subspace  $S \subseteq V$  satisfying  $A(S) \subseteq S$ ,  $B(S) \subseteq S$  and  $\text{im}(I) \subseteq S$ ;
- *co-stable* if there exists no nonzero subspace  $S \subseteq V$  satisfying  $A(S) \subseteq S$ ,  $B(S) \subseteq S$  and  $\ker(J) \supseteq S$ .

Denote by  $\mathbb{M}^s(r, n)$  (resp  $\mathbb{M}^c(r, n)$ ) the subset of  $\mathbb{M}(r, n)$  consisting of stable (resp. co-stable) ADHM data. They are open and invariant subsets. Call  $\mathbb{M}^{sc}(r, n)$  their intersection. The motivation to define such spaces lies in the following theorem.

**Theorem 4.3.** [Na, BM] *The  $GL(V)$  action is free and locally proper on  $\mathbb{M}^s(r, n)$ . Let  $\mathbb{M}^s/GL(V)$  be the associated geometric quotient. There exists an isomorphism  $\mathcal{M}_{\mathbb{P}^2}^l(r, n) \cong \mathbb{M}^s/GL(V)$ , where  $l \subseteq \mathbb{P}^2$  is a fixed line, which maps isomorphically  $\mathcal{M}^{reg}(r, n)$  onto  $\mathbb{M}^{sc}/GL(V)$ .  $\mathcal{M}(r, n)$  is a smooth connected quasi-projective variety of dimension  $2rn$ .*

*Remark 4.4.* We explain how to associate a framed sheaf to an ADHM data  $(A, B, I, J)$ . Fix homogeneous coordinates  $(x : y : z)$  on  $\mathbb{P}^2$ , so that  $l = \{z = 0\}$ . Define the two maps of coherent sheaves  $\alpha = \alpha(A, B, I, J)$  and  $\beta = \beta(A, B, I, J)$ :

$$\begin{aligned} \alpha : \mathcal{O}_{\mathbb{P}^2}(-1) \otimes V &\rightarrow \mathcal{O}_{\mathbb{P}^2} \otimes (V \oplus V \oplus W), \quad \alpha = (z \cdot A + x, z \cdot B + y, z \cdot J)^\top; \\ \beta : \mathcal{O}_{\mathbb{P}^2} \otimes (V \oplus V \oplus W) &\rightarrow \mathcal{O}_{\mathbb{P}^2}(1) \otimes V, \quad \beta = (-z \cdot B - y, z \cdot A + x, z \cdot I). \end{aligned}$$

The equation  $[A, B] + IJ = 0$  is indeed equivalent to  $\beta \circ \alpha = 0$ , and the stability and co-stability conditions for the datum correspond respectively to the surjectivity of  $\beta$  and injectivity of  $\alpha$  as a map of *vector bundles* (as a map of coherent sheaves,  $\alpha$  is automatically injective). Thus, a three-term complex of sheaves, exact anywhere but in degree 0, has been associated with an ADHM datum:

$$\mathcal{O}_{\mathbb{P}^2}(-1) \otimes V \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^2} \otimes (V^{\oplus 2} \oplus W) \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^2}(1) \otimes V.$$

Define  $E = \ker(\beta)/\text{im}(\alpha)$ . If one restricts the monad to  $D$  (i.e. imposes  $z = 0$ ), its cohomology defines the trivial bundle  $\mathcal{O}_D \otimes W$  on  $D$ . Thus, we get an induced isomorphism  $a : E|_D \rightarrow \mathcal{O}_D \otimes W$ . In addition, it turns out that any framed sheaf on  $\mathbb{P}^2$  can be realized as the cohomology of a complex as above, establishing an isomorphism as stated.

We shall see that it is possible to interpret framed symplectic sheaves by means of suitable modification of these ADHM data. We start with a definition.

**Definition 4.5.** We define a *monad*  $M$  on a scheme  $S$  to be a complex of locally free sheaves

$$\mathcal{U} \xrightarrow{\alpha} \mathcal{W} \xrightarrow{\beta} \mathcal{T}$$

with  $\alpha$  injective and  $\beta$  surjective. Let  $\mathcal{E}(M) = \ker(\beta)/\text{im}(\alpha)$ .

*Remark 4.6.* Let  $\mathcal{T}^\vee \xrightarrow{\beta^\vee} \mathcal{W}^\vee \xrightarrow{\alpha^\vee} \mathcal{U}^\vee$  be the dual complex  $M^\vee$ . This is no longer a monad since  $\alpha^\vee$  need not be surjective. Nevertheless, we have a natural isomorphism  $H^1(M^\vee) \cong \mathcal{E}^\vee$ . This

is proved using the so called *display of the monad*, as follows. From the monad  $M$  we construct a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \mathcal{U} & \xlongequal{\quad} & \mathcal{U} & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \ker(\beta) & \longrightarrow & \mathcal{W} & \longrightarrow & \mathcal{T} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & \mathcal{E} & \longrightarrow & \operatorname{coker}(\alpha) & \longrightarrow & \mathcal{T} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 
 \end{array}$$

Passing to duals, we get a diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \mathcal{T}^\vee & \longrightarrow & \operatorname{coker}(\alpha)^\vee & \longrightarrow & \mathcal{E}^\vee \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathcal{T}^\vee & \longrightarrow & \mathcal{W}^\vee & \longrightarrow & \ker(\beta)^\vee \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & \mathcal{U}^\vee & \xlongequal{\quad} & \mathcal{U}^\vee
 \end{array}$$

which gives  $\ker(\alpha^\vee)/\operatorname{im}(\beta^\vee) \cong \mathcal{E}^\vee$ .

In particular, for any morphism of complexes

$$\begin{array}{ccccc}
 \mathcal{U} & \longrightarrow & \mathcal{W} & \longrightarrow & \mathcal{T} \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathcal{T}^\vee & \longrightarrow & \mathcal{W}^\vee & \longrightarrow & \mathcal{U}^\vee
 \end{array}$$

we obtain a morphism  $\mathcal{E} \rightarrow \mathcal{E}^\vee$ .

The following proposition is a slight generalization of [OSS, Lemma 4.1.3].

**Proposition 4.7.** *Fix two complexes*

$$M: \mathcal{U} \xrightarrow{\alpha} \mathcal{W} \xrightarrow{\beta} \mathcal{T}$$

$$M' : \mathcal{U}' \xrightarrow{\alpha'} \mathcal{W}' \xrightarrow{\beta'} \mathcal{T}'$$

where  $M$  is a monad and  $\alpha'$  is injective. Let  $\mathcal{E} = H^1(M)$  and  $\mathcal{E}' = H^1(M')$ . Assume that the following vanishings hold:

$$\mathrm{Hom}_S(\mathcal{W}, \mathcal{U}') = \mathrm{Hom}_S(\mathcal{T}, \mathcal{W}') = 0$$

$$\mathrm{Ext}_S^1(\mathcal{T}, \mathcal{U}') = \mathrm{Ext}_S^1(\mathcal{W}, \mathcal{U}') = \mathrm{Ext}_S^1(\mathcal{T}, \mathcal{W}') = \mathrm{Ext}_S^2(\mathcal{T}, \mathcal{U}') = 0.$$

Then the natural morphism  $\mathrm{Hom}(M, M') \rightarrow \mathrm{Hom}_S(\mathcal{E}, \mathcal{E}')$  is a bijection.

*Proof.* Fix a morphism  $\phi : \mathcal{E} \rightarrow \mathcal{E}'$ . This gives a morphism  $\ker(\beta) \rightarrow \ker(\beta')/\mathrm{im}(\alpha') = \mathcal{E}'$ . From the exact sequence

$$\mathrm{Hom}(\ker(\beta), \mathcal{U}') \rightarrow \mathrm{Hom}(\ker(\beta), \ker(\beta')) \rightarrow \mathrm{Hom}(\ker(\beta), \mathcal{E}') \rightarrow \mathrm{Ext}^1(\ker(b), \mathcal{U}')$$

one deduces  $\mathrm{Hom}(\ker(\beta), \ker(\beta')) \cong \mathrm{Hom}(\ker(\beta), \mathcal{E}')$ , since  $\mathrm{Hom}(\ker(\beta), \mathcal{U}')$  and  $\mathrm{Ext}^1(\ker(b), \mathcal{U}')$  sit in an exact sequence between  $\mathrm{Hom}(\mathcal{W}, \mathcal{U}')$ ,  $\mathrm{Ext}^1(\mathcal{T}, \mathcal{U}')$  and  $\mathrm{Ext}^1(\mathcal{W}, \mathcal{U}')$ ,  $\mathrm{Ext}^2(\mathcal{T}, \mathcal{U}')$  respectively. These four spaces vanish by hypothesis. So,  $\phi$  lifts uniquely to a morphism

$$\phi_1 \in \mathrm{Hom}(\ker(\beta), \ker(\beta')) \subseteq \mathrm{Hom}(\ker(\beta), \mathcal{W}').$$

As  $\mathrm{Hom}(\mathcal{T}, \mathcal{W}') = \mathrm{Ext}^1(\mathcal{T}, \mathcal{W}') = 0$ , we get  $\mathrm{Hom}(\ker(\beta), \mathcal{W}') \cong \mathrm{Hom}(\mathcal{W}, \mathcal{W}')$ . This provides indeed an inverse for  $\mathrm{Hom}(M, M') \rightarrow \mathrm{Hom}_S(\mathcal{E}, \mathcal{E}')$ .  $\square$

*Remark 4.8.* Let  $(E, a, \varphi)$  be an  $l$ -framed symplectic sheaf on  $\mathbb{P}^2$ . Write  $E$  as the cohomology of a monad

$$\mathcal{O}(-1) \otimes V \xrightarrow{\alpha} \mathcal{O} \otimes (V^{\oplus 2} \oplus W) \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^2}(1) \otimes V$$

as in Remk. 4.4,  $\alpha = (z \cdot A + x, z \cdot B + y, z \cdot J)^\top$ ,  $\beta = (-z \cdot B - y, z \cdot A + x, z \cdot I)$  with  $(A, B, I, J) \in \mathbb{M}^s(r, n)$ . We note that this monad satisfies the hypothesis of Prop. 4.7, so that we can lift  $\varphi$  to a morphism of complexes

$$\begin{array}{ccccc} \mathcal{O}_{\mathbb{P}^2}(-1) \otimes V & \xrightarrow{\alpha} & \mathcal{O}_{\mathbb{P}^2} \otimes (V \oplus V \oplus W) & \xrightarrow{\beta} & \mathcal{O}_{\mathbb{P}^2}(1) \otimes V \\ G_1 \downarrow & & F \downarrow & & \downarrow G_2 \\ \mathcal{O}_{\mathbb{P}^2}(-1) \otimes V^\vee & \xrightarrow{\beta^\vee} & \mathcal{O}_{\mathbb{P}^2} \otimes (V^\vee \oplus V^\vee \oplus W^\vee) & \xrightarrow{\alpha^\vee} & \mathcal{O}_{\mathbb{P}^2}(1) \otimes V^\vee \end{array}$$

*Remark 4.9.* It is proved in [JMW] that the commutativity of this diagram, with some additional conditions on  $\varphi$  (namely, skew-symmetry and compatibility with the framing), are equivalent to the following set of conditions:

- $G_2 = -G_1 := G$ ;
- $F = \begin{pmatrix} 0 & G & 0 \\ -G & 0 & 0 \\ 0 & 0 & \Omega \end{pmatrix}$ ;
- $GA - A^\vee G = 0 = GB - B^\vee G$ ;

- $J = -\Omega^{-1}I^\vee G$ .

This motivates the following definition.

**Definition 4.10.** The *variety of symplectic ADHM data* of type  $(r, n)$  is the closed subvariety of the affine space

$$\mathbb{M}_\Omega(r, n) \subseteq \text{End}(V)^{\oplus 2} \oplus \text{Hom}(W, V) \oplus \text{Hom}(S^2 V, \mathbb{C})$$

defined by the equations

- $GA - A^\vee G = 0$  (*GA-symmetry*);
- $GB - B^\vee G = 0$  (*GB-symmetry*);
- $[A, B] - I\Omega^{-1}I^\vee G = 0$  (*ADHM equation*).

The group  $GL(V)$  acts on  $\mathbb{M}_\Omega(r, n)$  naturally:

$$g \cdot (A, B, I, G) = (gAg^{-1}, gBg^{-1}, gI, g^{-\vee}Gg^{-1}).$$

To any symplectic ADHM datum one associates a “classic” datum, by defining  $J = -\Omega^{-1}I^\vee G$ :

$$\iota : \mathbb{M}_\Omega(r, n) \rightarrow \mathbb{M}(r, n), \quad \iota(A, B, I, G) = (A, B, I, -\Omega^{-1}I^\vee G).$$

The map  $\iota$  is clearly a  $GL(V)$ -equivariant morphism. We call a symplectic datum stable or co-stable if its associated classic ADHM datum is, and we denote  $\mathbb{M}_\Omega^s(r, n)$ ,  $\mathbb{M}_\Omega^c(r, n)$  and  $\mathbb{M}_\Omega^{sc}(r, n)$  the corresponding open invariant subsets.

**Lemma 4.11.** *Let  $(A, B, I, G)$  be a stable symplectic datum. If  $S \subseteq V$  is an  $A, B$ -invariant subspace satisfying  $\ker(-\Omega^{-1}I^\vee G) \subseteq S$ , then  $S \subseteq G$ . In particular,  $(A, B, I, G)$  is co-stable if and only if  $G$  is invertible.*

*Proof.* Let  $S \subseteq \ker(-\Omega^{-1}I^\vee G) \subseteq V$  be an  $A, B$ -stable subspace. Let  $s \in S$ ,  $G(s) \in V^\vee$ . Then  $G(s)^\perp \supseteq \text{Im}(I)$ . Let

$$T = \bigcap_{s \in S} G(s)^\perp \subseteq V.$$

Using the  $GA$ -symmetry, we prove  $T$  is  $A$ -stable:

$$\begin{aligned} t \in T &\implies \langle G(s), A(t) \rangle = \langle A^\vee G(s), t \rangle = \\ &= \langle GA(s), t \rangle = 0, \end{aligned}$$

since  $A(s) \in S$ . The same holds for  $B$ . The stability of the datum forces  $T = V$ ; but this means  $G(S) = 0$ , i.e.  $S \subseteq \text{Ker}(G)$ .  $\square$

**4.2. ADHM type description for  $\mathcal{M}_{\mathbb{P}^2, \Omega}^l$ .** The aim of this section is to define a scheme structure on the set of equivalence classes of symplectic ADHM configurations. We shall also give a symplectic analogue to Theorem 4.3, i.e. we will prove that the resulting scheme is in fact isomorphic to the moduli space of framed symplectic sheaves on the plane. The scheme structure is constructed by means of the following lemma.

**Lemma 4.12.** *The restriction of the map  $\iota$  to  $\mathbb{M}_\Omega^s$  is a closed embedding.*

*Proof.* We want to apply Lemma 2.5. First we verify that if  $(A, B, I, G_1)$  and  $(A, B, I, G_2)$  are stable symplectic data satisfying  $I^\vee G_1 = I^\vee G_2$ , then  $G_1 = G_2$ . Let  $S = \text{Ker}(G_1 - G_2)$ ; it is  $A, B$ -stable by the  $GA$  and  $GB$  symmetries. Moreover,

$$(G_1 - G_2) \circ I = [I^\vee(G_1^\vee - G_2^\vee)]^\vee = [I^\vee(G_1 - G_2)]^\vee = 0,$$

i.e.  $S \supseteq \text{Im}(I)$ . This forces  $S = V$ , and proves injectivity at closed points.

The tangent space  $T_{(A,B,I,GJ)}\mathbb{M}^s$  can be identified with the vector space of quadruples

$$(X_A, X_B, X_I, X_J) \in \text{End}(V)^{\oplus 2} \oplus \text{Hom}(W, V) \oplus \text{Hom}(V, W)$$

satisfying the equation

$$[A, X_B] + [X_A, B] + X_I J - I X_J = 0.$$

The corresponding description for  $T_{(A,B,I,G)}\mathbb{M}_\Omega^s$  is:

$$(X_A, X_B, X_I, X_G) \in \text{End}(V)^{\oplus 2} \oplus \text{Hom}(W, V) \oplus \text{Hom}(S^2 V, \mathbb{C})$$

satisfying the equations

- $G X_A - X_A^\vee G + X_G A - A^\vee X_G = 0$ ;
- $G X_B - X_B^\vee G + X_G B - B^\vee X_G = 0$ ;
- $[A, X_B] + [X_A, B] - X_I \Omega^{-1} I G - I \Omega^{-1} X_I G - I \Omega^{-1} I X_G = 0$ .

We write the tangent map:

$$T_{(A,B,I,G)}\mathbb{M}_\Omega^s \ni (X_A, X_B, X_I, X_G) \mapsto (X_A, X_B, X_I, -\Omega^{-1} X_I^\vee G - \Omega^{-1} I^\vee X_G) \in T_{(A,B,I,J)}\mathbb{M}^s.$$

Suppose  $(X_A, X_B, X_I, X_G) \mapsto 0$ . This forces  $X_G A - A^\vee X_G, X_G B - B^\vee X_G = 0$  and  $I^\vee X_G = 0$ , and we may conclude  $(X_A, X_B, X_I, X_G) = 0$  by following the same argument we employed to prove injectivity on closed points (i.e. proving that  $\ker(G)$  is an invariant subspace of  $V$  containing  $I(W)$ ).

Eventually, we need to prove properness; we will apply a valuative criterion as stated in [GD, 7.3.9]. Let  $\text{Spec}(\mathbb{C}((t))) \rightarrow \text{Spec}(\mathbb{C}[[t]])$  be the inclusion of the pointed one-dimensional formal disc into the formal disc. If for every commutative diagram

$$\begin{array}{ccc} \text{Spec}(\mathbb{C}((t))) & \longrightarrow & \mathbb{M}_\Omega^s \\ \downarrow & \nearrow & \downarrow \\ \text{Spec}(\mathbb{C}[[t]]) & \longrightarrow & \mathbb{M}^s \end{array}$$

we are able to find a lifting as above, then properness is proved. This can be rephrased as follows. Suppose  $(A_t, B_t, I_t, J_t) \in \mathbb{M}(\mathbb{C}[[t]])$  such that the pullback to  $\text{Spec}(\mathbb{C})$ , denoted  $(A_0, B_0, I_0, J_0)$ , gives a stable point, and let  $G_t$  be a symmetric matrix with entries in  $\mathbb{C}((t))$  for which  $(A_t, B_t, I_t, G_t) \in \mathbb{M}_\Omega(\mathbb{C}((t)))$  and  $J_t = -\Omega^{-1} I_t^\vee G_t$ . We need to prove that the entries of  $G_t$  sit in fact in  $\mathbb{C}[[t]]$ . Suppose not. It follows that  $G_t \neq 0$ , and one can write  $G_t = G_0 t^{-k} + t^{-k+1} G_1 + \dots$  with  $G_i$  symmetric  $\mathbb{C}$ -matrices,  $G_0 \neq 0$  and  $k$  a positive integer.

We can change the coordinates and put  $G_0$  in the form

$$G_0 = \begin{pmatrix} \mathbb{1}_m & 0 \\ 0 & 0_{n-m} \end{pmatrix}, m \neq 0.$$

Write  $A_t = A_0 + tA_1 + \dots$ . The term of order  $-k$  in the equation  $G_t A_t - A_t^\vee G_t = 0$  gives  $G_0 A_0 - A_0^\vee G_0 = 0$ , which means that the matrix  $A_0$  is of the form

$$A_0 = \begin{pmatrix} \star & 0 \\ \star & \star \end{pmatrix};$$

and the same holds for  $B_0$ . Write now  $I_t = I_0 + tI_1 + \dots$ , and let

$$I_0 = \begin{pmatrix} I_0^1 \\ I_0^2 \end{pmatrix}$$

be the block decomposition coherent with our choice of coordinates. Since  $J_t = -\Omega^{-1} I_t^\vee G_t$  has entries in  $\mathbb{C}[[t]]$ , we get  $I_0^\vee G_0 = 0$  as above. This implies  $I_0^1 = 0$ , and contradicts the stability of the quadruple  $(A_0, B_0, I_0, J_0)$  since the space of vectors of type  $\begin{pmatrix} 0 \\ v \end{pmatrix}$  form an  $A_0, B_0$  invariant subspace containing in the image of  $I_0$ .  $\square$

*Remark 4.13.* As a consequence of this lemma, we obtain a scheme structure on the set  $\mathbb{M}_\Omega^s/GL(V)$ . Indeed, we have that  $\mathbb{M}_\Omega^s$  is a closed  $GL(V)$ -invariant subscheme of  $\mathbb{M}^s$ , and since  $\mathbb{M}^s/GL(V)$  exists as a geometric quotient as the action is free and locally proper, we can conclude that the same holds for  $\mathbb{M}_\Omega^s/GL(V)$ . Furthermore, the projection  $\mathbb{M}_\Omega^s \rightarrow \mathbb{M}_\Omega^s/GL(V)$  is a  $GL(V)$ -principal bundle (as it is the pullback of the bundle  $\mathbb{M}^s \rightarrow \mathbb{M}^s/GL(V)$  by definition).

The last tool we need to establish the desired isomorphism of moduli spaces is the universal monad on  $\mathcal{M}_{\mathbb{P}^2}^l(r, n)$ .

**Lemma 4.14.** [He, Sect. 7] *There exists a monad*

$$\mathcal{U} \xrightarrow{\alpha} \mathcal{V} \xrightarrow{\beta} \mathcal{T}$$

on  $\mathbb{P}^2 \times \mathcal{M}_{\mathbb{P}^2}^l(r, n)$  which is universal in the following sense: the cohomology of the monad and its restriction to  $l \times \mathcal{M}_{\mathbb{P}^2}^l(r, n)$  give a pair  $(\mathcal{E}, a)$  which is a universal framed sheaf for the moduli space. Let  $V, W$  be as in Def. 4.1. There exists an open affine cover  $\{U_i\}$  of  $\mathcal{M}_{\mathbb{P}^2}^l(r, n)$  satisfying the following properties:

- the principal  $GL(n)$  bundle  $\mathbb{M}^s(r, n) \rightarrow \mathcal{M}_{\mathbb{P}^2}^l(r, n)$  is trivial on  $U_i$ ;
- the restriction of the monad to  $\mathbb{P}^2 \times U_i$  looks like

$$\mathcal{O}(-1) \boxtimes \mathcal{O}_{U_i} \otimes V \xrightarrow{\alpha_i} \mathcal{O} \boxtimes \mathcal{O}_{U_i} \otimes (V^{\oplus 2} \oplus W) \xrightarrow{\beta_i} \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{U_i} \otimes V$$

and one can choose  $\alpha_i$  and  $\beta_i$  as follows: let  $\sigma_i : U_i \rightarrow \mathbb{M}^s$  be a section, and define

$$\alpha_i = \alpha \circ \sigma_i, \beta_i = \beta \circ \sigma_i,$$

see *Remk 4.4*.

We are finally ready to state and prove the main theorem of this section.

**Theorem 4.15.** *There exists an isomorphism of schemes  $\mathcal{M}_{\mathbb{P}^2, \Omega}^l \cong \mathbb{M}_{\Omega}^s/GL(V)$  which maps  $\mathcal{M}_{\mathbb{P}^2, \Omega}^{l, reg}$  to  $\mathbb{M}_{\Omega}^{sc}/GL(V) = \{\det(G) \neq 0\}/GL(V)$ .*

*Proof.* Let  $(\mathcal{E}_S, \chi_S, \varphi_S)$  be an  $S$ -point of  $\mathcal{M}_{\mathbb{P}^2, \Omega}^l$  for  $S$  a given scheme, i.e. an  $S$ -flat family of framed symplectic sheaves on  $\mathbb{P}^2$ . The pair  $(\mathcal{E}_S, \chi_S)$  induces a morphism  $S \rightarrow \mathbb{M}^s/GL(V) \cong \mathcal{M}_{\mathbb{P}^2}^l$ . Let

$$M : \mathcal{U} \rightarrow \mathcal{W} \rightarrow \mathcal{T}$$

be a universal monad on  $\mathcal{M}_{\mathbb{P}^2}^l \times \mathbb{P}^2$ ; if we call  $M_S$  its pullback to  $S \times \mathbb{P}^2$ , we can write by definition  $\mathcal{E}_S = H^1(M_S)$ . Choose an affine open cover  $\{S_i\}$  of  $S$  so that the monad  $M_{S_i}$  is of the form

$$(\mathcal{O}_{\mathbb{P}^2}(-1) \boxtimes \mathcal{O}_{S_i}) \otimes V \rightarrow (\mathcal{O}_{\mathbb{P}^2} \boxtimes \mathcal{O}_{S_i}) \otimes (V^{\oplus 2} \oplus W) \rightarrow (\mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{S_i}) \otimes V.$$

(it is enough to choose an affine refinement of the open cover given by the preimage of a suitable open cover  $\{U_i\}$ , see Lemma 4.14). Consider now  $\varphi_{S_i} : H^1(M_{S_i}) \cong \mathcal{E}_{S_i} \rightarrow \mathcal{E}_{S_i}^\vee \cong H^1(M_{S_i}^\vee)$ . Since the pair  $(M_{S_i}, M_{S_i}^\vee)$  satisfies the hypothesis of Prop. 4.7, we can lift uniquely  $\varphi_{S_i}$  to a morphism of complexes

$$M_{S_i} \rightarrow M_{S_i}^\vee,$$

and if we apply the constraint on the framing and skew-symmetry of  $\varphi_S$  as in 4.9, we get a morphism  $f_i : S_i \rightarrow \mathbb{M}_{\Omega}^s$ . On the overlaps  $S_{ij}$  we have  $f_i \sim f_j$  under the natural action of  $GL(V)$ ; therefore, we obtain a map  $S \rightarrow \mathbb{M}_{\Omega}^s/GL(V)$ . This defines a map  $\mathcal{M}_{\mathbb{P}^2, \Omega}^l \rightarrow \mathbb{M}_{\Omega}^s/GL(V)$ .

We explain how to construct an inverse for this map. Let  $(\underline{\mathcal{E}}, \underline{\alpha})$  be a universal framed sheaf on  $\mathcal{M}_{\mathbb{P}^2, \Omega}^l \times \mathbb{P}^2$ , and call  $(\mathcal{E}, \alpha)$  its restriction to the closed subscheme  $\mathbb{M}_{\Omega}^s/GL(V)$ . We want to construct a symplectic form  $\Phi$  on  $\mathcal{E}$ . Choose an open cover of  $\mathbb{M}^s/GL(V)$  as in Lemma 4.14, and define  $\{U_i\}$  to be the induced open cover of the closed subscheme  $\mathbb{M}_{\Omega}^s/GL(V)$ . The following conditions hold simultaneously:

- (1) the pullback of the universal monad to  $U_i \times \mathbb{P}^2$  is of the form

$$(\mathcal{O}_{\mathbb{P}^2}(-1) \boxtimes \mathcal{O}_{U_i}) \otimes V \rightarrow (\mathcal{O}_{\mathbb{P}^2} \boxtimes \mathcal{O}_{U_i}) \otimes (V^{\oplus 2} \oplus W) \rightarrow (\mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{U_i}) \otimes V;$$

- (2) the principal bundle  $\mathbb{M}_{\Omega}^s \rightarrow \mathbb{M}_{\Omega}^s/GL(V)$  is trivialized on  $U_i$ .

We fix sections  $\sigma_i : U_i \rightarrow \mathbb{M}_{\Omega}^s$ . Write  $\sigma_i = (A_i, B_i, I_i, G_i)$ . We obtain a morphism  $G_i : U_i \rightarrow \text{Hom}_{\mathbb{C}}(V, V^\vee)$ , which we use to define a diagram

$$\begin{array}{ccccc} (\mathcal{O}_{\mathbb{P}^2}(-1) \boxtimes \mathcal{O}_{U_i}) \otimes V & \longrightarrow & (\mathcal{O}_{\mathbb{P}^2} \boxtimes \mathcal{O}_{U_i}) \otimes (V^{\oplus 2} \oplus W) & \longrightarrow & (\mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{U_i}) \otimes V \\ \downarrow -G_i & & \downarrow \Omega_i & & \downarrow G_i \\ (\mathcal{O}_{\mathbb{P}^2}(-1) \boxtimes \mathcal{O}_{U_i}) \otimes V^\vee & \longrightarrow & (\mathcal{O}_{\mathbb{P}^2} \boxtimes \mathcal{O}_{U_i}) \otimes (V^{\vee \oplus 2} \oplus W^\vee) & \longrightarrow & (\mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{U_i}) \otimes V^\vee \end{array}$$



where

$$\Omega_i = \begin{pmatrix} 0 & G_i & 0 \\ -G_i & 0 & 0 \\ 0 & 0 & \Omega \end{pmatrix}.$$

This yields a morphism of complexes by construction and induces a collection of morphisms  $\varphi_i : \mathcal{E}|_{U_i} \rightarrow \mathcal{E}|_{U_i}$  which are  $GL(V)(S_i)$ -equivalent on the overlaps. We defined a  $\mathbb{M}_\Omega^s/GL(V)$ -family of framed symplectic sheaves on  $\mathbb{P}^2$ , i.e. a morphism  $\mathbb{M}_\Omega^s/GL(V) \rightarrow \mathcal{M}_{\mathbb{P}^2, \Omega}^l$ . The statement about the locally free locus is a direct consequence of Lemma 4.11.  $\square$

## 5. IRREDUCIBILITY OF $\mathcal{M}_{\mathbb{P}^2, \Omega}^l$

The aim of the section is to prove the irreducibility of the moduli space of framed symplectic sheaves  $\mathcal{M}_{\mathbb{P}^2, \Omega}^l(r, n) =: \mathcal{M}_\Omega(r, n)$ . We will make use of its description as the orbit space of the action of  $GL(V)$  on the space of stable symplectic ADHM configurations  $\mathbb{M}_\Omega^s(r, n)$ , as explained in the previous section. We fix Darboux coordinates on the symplectic vector space  $(W, \Omega)$  so that  $-\Omega^{-1} = \Omega$ ,  $W \cong \mathbb{C}^r$ . We recall that the space of quadruples  $(A, B, I, G)$  whose orbits correspond to symplectic bundles are the ones belonging to the open invariant subset  $\{rk(G) = n\}$ .

*Remark 5.1.* The double dual of a symplectic sheaf is a symplectic bundle with  $c_2 = rk(G)$ . We will show how to extract an ADHM datum for  $E^{\vee\vee}$  from  $E = [A, B, I, G]$ . Fix coordinates so that  $G = \begin{pmatrix} \mathbb{1}_{rk(G)} & 0 \\ 0 & 0_{n-rk(G)} \end{pmatrix}$ . The  $GA, GB$  symmetries imply that one has

$$A = \begin{pmatrix} A' & 0 \\ a & \alpha \end{pmatrix}, B = \begin{pmatrix} B' & 0 \\ b & \beta \end{pmatrix}$$

with  $A'$  and  $B'$  symmetric. Write  $I = \begin{pmatrix} I' \\ X \end{pmatrix}$  according to the above decomposition. We obtain the symplectic ADHM quadruple  $(A', B', I', \mathbb{1}_{rk(G)})$  which is a representative for the point  $[E^{\vee\vee}] \in \mathcal{M}_\Omega^{reg}(r, rk(G))$ .

**Strategy of the proof.** The locally free locus  $\mathcal{M}_\Omega^{reg}(r, n)$  is smooth of dimension  $rn + 2n$  and connected, see for example [BFG]; it is then an irreducible variety. The idea is to prove that the closure of this open subset coincides with the entire  $\mathcal{M}_\Omega$ . For any given  $(A, B, I, G)$ , we shall provide a rather explicit construction of a rational curve in the moduli space passing through  $[A, B, I, G]$  and whose general point lies in  $\mathcal{M}_\Omega^{reg}$ . We shall start studying the cases  $G = 0$  and  $rk(G) = n - 1$ , as in the proof for the general case we will use a blend of the techniques for these two extremal cases.

**5.1. The case  $G = 0$ .** The techniques for this set up are largely inspired from [Ba1, Appendix A].

**Definition 5.2.** A matrix  $A \in M_n(\mathbb{C})$  is said to be cyclic or nonderogatory if its minimal polynomial is equal to the characteristic polynomial; equivalently, there exists a vector  $v \in \mathbb{C}^n$  such that  $v, Av, \dots, A^{n-1}v$  span  $\mathbb{C}^n$ . Clearly, the transpose of a nonderogatory matrix is again nonderogatory. Further characterizations of these matrices:

- $A$  is nonderogatory if and only if any matrix commuting with  $A$  is a polynomial in  $A$ , i.e.

$$[A, B] = 0 \implies \exists P \in \mathbb{C}[t] \mid B = P(A).$$

- $A$  is nonderogatory if and only if all of its eigenvalues have geometric multiplicity equal to 1 (ex: matrices with no repeated eigenvalues, Jordan blocks).

The space of nonderogatory matrices is a nonempty open subset of  $M_n$ .

We quote now the main result in [TZ]:

**Theorem 5.3.** *For any fixed matrix  $A$  there exists a nonsingular symmetric matrix  $g$  such that  $gAg^{-1} = A^\top$ . Any matrix  $g$  transforming  $A$  into its transpose is symmetric if and only if  $A$  is nonderogatory.*

A consequence of this is that any complex matrix  $A$  is similar to a symmetric one. Let  $gAg^{-1} = A^\top$  with  $g$  symmetric. Write  $g = s \cdot s^\top$  with  $s$  nonsingular. This gives

$$s^\top As^{-\top} = s^{-1}A^\top s = (s^\top As^{-\top})^\top.$$

As an immediate corollary one gets:

**Corollary 5.4.** *Suppose  $(A, B)$  is a pair of commuting matrices, and suppose  $A$  is nonderogatory. There exists a nonsingular matrix  $g$  such that  $gAg^{-1}$  and  $gBg^{-1}$  are symmetric.*

We need another useful result:

**Proposition 5.5.** *Let  $B$  be any matrix. There exists a nonderogatory matrix  $N$  commuting with  $B$ .*

*Proof.* Put  $B$  in Jordan form. Perturb each Jordan block by small multiples of the identity, so that any two distinct blocks are relative to distinct eigenvalues. The output of this construction is matrix commuting with  $B$ , whose eigenvalues all have geometric multiplicity equal to 1.  $\square$

Consider now a framed symplectic sheaf  $E$  represented by an ADHM quadruple  $(A, B, I, 0)$ . Consequently, the ADHM equation reduces to  $[A, B] = 0$ , and the symmetries are vacuous. Suppose  $A$  is nonderogatory. One can find a basis for which both  $A$  and  $B$  are symmetric by Cor. 5.4.

Denote the linear subspaces of  $M_n$  of symmetric and antisymmetric matrices respectively  $S_n$  and  $AS_n$ . The linear map

$$[A, \_ ] : S_n \rightarrow AS_n$$

is surjective. Indeed, its kernel has dimension exactly  $n = \dim(S_n) - \dim(AS_n)$ , which is the dimension of its subspace defined by polynomials in  $A$ .

As a consequence, we can find a symmetric matrix  $X$  so that  $[A, X] = I\Omega I^\perp$ . For  $t \in \mathbb{C}$ , consider the family of quadruples  $(A, B + tX, I, t \cdot Id)$ . Since  $A$ ,  $B$  and  $X$  are symmetric, the  $GA$ ,  $GB$  symmetry equations are satisfied and, by construction, the  $ADHM$  equation is satisfied as well. For small values of  $t$  we can preserve stability, as it is an open condition. Finally, as  $t \cdot Id$  is invertible for  $t \neq 0$ , we found that  $E = [(A, B, I, 0)]$  sits in the closure of  $\mathcal{M}_\Omega^{reg}$ .

We need to show now that the  $A$ -cyclicity hypothesis may be dropped. Let  $(A, B, I, 0)$  as above, and let  $N$  be a nonderogatory matrix with  $[N, B] = 0$ . Consider the line

$$A_t = (1 - t)A + tN, \quad t \in \mathbb{C}.$$

This line contains  $N$ , and since the set of nonderogatory matrices is open, one must have that  $A_t$  is nonderogatory for any  $t$ , except a finite number of values. Furthermore, by construction  $[A_t, B] = 0$ . We proved that every neighborhood of  $[(A, B, I, 0)]$  contains points of  $\overline{\mathcal{M}_\Omega^{reg}}$ ; this tells us that  $[(A, B, I, 0)] \in \overline{\mathcal{M}_\Omega^{reg}}$ .

**5.2. The case  $rk(G) = n - 1$ .** These ADHM data correspond to symplectic sheaves whose singular locus is concentrated in one point, with multiplicity 1. We can choose coordinates such that

$$G = \begin{pmatrix} \mathbb{1}_{n-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

The other matrices can be written as

$$\begin{pmatrix} A & 0 \\ a & \alpha \end{pmatrix}, \begin{pmatrix} B & 0 \\ b & \beta \end{pmatrix}, \begin{pmatrix} I \\ X \end{pmatrix}.$$

with  $A$  and  $B$  symmetric. Without loss of generality we may assume  $\alpha = \beta = 0$ , since  $\mathbb{A}^2$  acts on the space of ADHM configurations by adding multiples of the identity matrix to the endomorphisms. Of course, the choice of  $a$  and  $b$  is not unique: by changing the coordinates we can replace them respectively with  $vA + a\lambda$ ,  $vB + b\lambda$  for a nonzero  $\lambda \in \mathbb{C}$ , keeping  $A, B, I$  fixed (the transformation  $g = \begin{pmatrix} \mathbb{1}_{n-1} & 0 \\ v & \lambda \end{pmatrix}$  does the job).

*Remark 5.6.* The subspace of  $\mathbb{C}^{n-1}$

$$S = im(A) + im(I) + im(BI) + im(B^2I) + \cdots + im(B^{n-2}I)$$

coincides in fact with the whole  $\mathbb{C}^{n-1}$ . Indeed, the triple  $(A, B, I)$  is stable, and the subspace we are considering contains by definition the image of  $I$ , and it is  $A, B$ -invariant.  $A$ -invariance is obvious since  $S$  contains the image of  $A$ . To prove  $B$ -invariance, we first note that the subspace

$$im(I) + im(BI) + im(B^2I) + \cdots + im(B^{n-2}I)$$

is  $B$ -invariant (as  $B^{n-1}$  can be written as a polynomial in  $B$  of degree  $n - 2$  at most). Moreover,

$$B(Av) = A(Bv) + I(\Omega I^\top v) \in im(A) + im(I)$$

by  $[A, B] - I\Omega I^\top = 0$ .

*Remark 5.7.* If we can choose  $a$  (or  $b$ ) to be 0, then we can do the following. Consider the family of configurations

$$\left( \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} B & tb \\ b & 0 \end{pmatrix}, I, \begin{pmatrix} \mathbb{I}_{n-1} & 0 \\ 0 & t \end{pmatrix} \right).$$

(or the analogous one with  $b = 0$ ). This gives a locally free deformation of the sheaf  $E$ . This condition is verified, for example, when  $A$  (or  $B$ ) is invertible; in this case, we can write  $vA = a$  (or  $vB = b$ ) and change the coordinates accordingly.

We need the following technical lemma.

**Lemma 5.8.** *Let  $R$  be a  $n$ -dimensional vector spaces, let  $T \in \text{End}_{\mathbb{C}}(R)$  and  $L \subseteq R$  be a subspace. Let  $v \in R$  be  $T$ -reachable from  $L$ , meaning*

$$v \in L + TL + T^2L + \cdots + T^{n-1}L.$$

*There exist parameterized curves  $r : \mathbb{C} \rightarrow R$  and  $l : \mathbb{C} \rightarrow L$  with  $r(0) = 0$ ,  $l(0) = 0$  satisfying*

$$(T - t \cdot \text{Id}_R)r(t) = t \cdot v + l(t).$$

*Proof.* Write  $v = \sum_{i=0}^{n-1} T^i l_i$ ,  $l_i \in L$ . Define

$$r(t) = \sum_{i=1}^{n-1} t^i \left( \sum_{j=i}^{n-1} T^{j-i} l_j \right).$$

We obtain

$$\begin{aligned} (T - t\text{Id}_R)r(t) &= \sum_{i=1}^{n-1} t^i \left( \sum_{j=i}^{n-1} T^{j-i+1} l_j \right) - \sum_{i=1}^{n-1} t^{i+1} \left( \sum_{j=i}^{n-1} T^{j-i} l_j \right) = \\ &= t \cdot v - tl_0 + \sum_{i=2}^{n-1} t^i \left( \sum_{j=i}^{n-1} T^{j-i+1} l_j \right) - \sum_{i=1}^{n-2} t^{i+1} \left( \sum_{j=i+1}^{n-1} T^{j-i} l_j \right) - \sum_{i=1}^{n-2} t^{i+1} l_i = \\ &= t \cdot v - \sum_{i=0}^{n-1} t^{i+1} l_i + \sum_{i=2}^{n-1} t^i \left( \sum_{j=i}^{n-1} T^{j-i+1} l_j \right) - \sum_{k=2}^{n-1} t^k \left( \sum_{j=k}^{n-1} T^{j-k+1} l_k \right) = \\ &= t \cdot v - \sum_{i=0}^{n-1} t^{i+1} l_i. \end{aligned}$$

So, it is enough to set

$$l(t) = - \sum_{i=0}^{n-1} t^{i+1} l_i.$$

□

We apply the lemma to the following situation. Given an ADHM quadruple

$$\left( \begin{pmatrix} A & 0 \\ a & 0 \end{pmatrix}, \begin{pmatrix} B & 0 \\ b & 0 \end{pmatrix}, \begin{pmatrix} I \\ X \end{pmatrix}, \begin{pmatrix} \mathbb{1}_{n-1} & 0 \\ 0 & 0 \end{pmatrix} \right),$$

we set:  $R = \mathbb{C}^{n-1}$ ,  $L = \text{im}(I)$ ,  $v = a^\top$ . By Remk. 5.6 we can write

$$v = Av_A + Ix_0 + BIx_1 + \cdots + B^{n-1}Ix_{n-1}$$

for some vector  $v_A \in R$  and  $x_i \in W$ , and we can find an equivalent triple with the same  $A$ ,  $B$  and  $I$  so that  $v = Ix_0 + BIx_1 + \cdots + B^{n-1}Ix_{n-1}$  (just remember we can move  $a$  by any vector in the image of  $A$ ). Let  $r(t)$  and  $l(t) = I(Y(t)) \in \text{im}(I)$  satisfying the thesis, and write the deformation

$$\left( \begin{pmatrix} A & 0 \\ a + r(t)^\top & 0 \end{pmatrix}, \begin{pmatrix} B - tId & 0 \\ b & 0 \end{pmatrix}, \begin{pmatrix} I \\ X + Y^\top(t) \cdot \Omega^{-1} \end{pmatrix}, \begin{pmatrix} \mathbb{1}_{n-1} & 0 \\ 0 & 0 \end{pmatrix} \right).$$

Now, any point of this curve sits in the space of ADHM configurations, because  $[A, B - tId] = [A, B]$ , the  $GA$ ,  $GB$  symmetries are obviously satisfied, and the (2,1) block of the commutator is written as

$$\begin{aligned} (a + r(t)^\top)(B - tId) - bA &= aB - bA - bA + r(t)^\top(B - tId) - ta = X\Omega I^\top + ta + Y^\top(t) \cdot I^\top - ta = \\ &= (X + Y^\top(t)\Omega^{-1})\Omega I^\top. \end{aligned}$$

The previous calculation exhibits a small deformation of the given configuration which has an invertible matrix  $(B - tId)$  in the (1,1) block, and  $\beta$  in the (2,2) entry: this must sit in  $\overline{\mathcal{M}}_\Omega^{reg}$  by Remk 5.7.

**5.3. The general case.** We are ready to deal with the case of quadruples  $(A', B', I', G)$  with  $k = rk(G) \in \{1, \dots, n-2\}$ , where  $n = \dim(V)$  as usual. We can normalize  $G$  to

$$G = \begin{pmatrix} \mathbb{1}_k & 0 \\ 0 & 0_{n-k} \end{pmatrix}$$

and thus write

$$A' = \begin{pmatrix} A & 0 \\ a & \alpha \end{pmatrix}, B' = \begin{pmatrix} B & 0 \\ b & \beta \end{pmatrix}, I' = \begin{pmatrix} I \\ X \end{pmatrix},$$

with  $A, B$  symmetric,  $[A, B] = I\Omega I^\top$ ,  $[\alpha, \beta] = 0$  and  $(aB - \beta a) - (bA - \alpha b) = X\Omega I^\top$ . We note that acting by the  $G$ -preserving transformation  $g_v = \begin{pmatrix} 1_k & 0 \\ v & 1_{n-k} \end{pmatrix}$  we leave  $A, B, \alpha, \beta$  and  $I$  untouched and move  $a$  and  $b$  respectively to  $a + vA - \alpha v$  and  $b + vB - \beta v$ . In order to deform our quadruple into a rank  $n$  one, we shall need once again to prove that we can slightly deform it and get a quadruple with vanishing  $a$  or  $b$ . In the  $n-1$  case, this was guaranteed from  $A$  or  $B$  being invertible if we have  $\alpha = \beta = 0$ . For general  $\alpha$  and  $\beta$ , what we should require is  $\alpha$  not an eigenvalue for  $A$  (or similarly for  $\beta$  and  $B$ ). More generally:

**Lemma 5.9.** *Let  $S \in \text{Mat}(k \times k)$  be symmetric and  $\sigma \in \text{Mat}((n-k) \times (n-k))$ . Suppose that  $S$  and  $\sigma$  share no eigenvalues. Then the linear map*

$$T \in \text{End}(\text{Mat}((n-k) \times k)), \quad T(v) = vS - \sigma v$$

*is invertible.*

*Proof.* Choose coordinates on  $\mathbb{C}^{n-k}$  so that  $\sigma$  is lower triangular:

$$\sigma = \begin{pmatrix} s_1 & 0 & \cdots \\ \star & s_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Suppose there exists a matrix  $v = \begin{pmatrix} v_1 \\ \vdots \\ v_{n-k} \end{pmatrix}$ ,  $v_i \in \mathbb{C}^k$  satisfying  $vS = \sigma v$ . We have  $\sigma v =$

$$\begin{pmatrix} s_1 v_1 \\ \star \\ \star \end{pmatrix} = vS = \begin{pmatrix} v_1 S \\ \vdots \\ v_{n-k} S \end{pmatrix}. \text{ We obtain } S(v_1^\top) = s_1 v_1^\top. \text{ So if } S \text{ and } \sigma \text{ have disjoint spectra, } T$$

must be injective, i.e. an isomorphism.  $\square$

We want to apply Lemma 5.8 again to prove that we can perturb the quadruple to separate the spectra of  $B$  and  $\beta$  in order to obtain a quadruple for which  $b = 0$ . We need the following generalization of Remk. 5.6.

**Lemma 5.10.** *Let  $R = \text{Mat}((n-k) \times k)$  and  $A, B, I$  as above ( $A$  and  $B$  symmetric  $k \times k$  matrices,  $I \in \text{Hom}(\mathbb{C}^r, \mathbb{C}^k)$  with  $(\mathbb{C}^r, \Omega)$  a symplectic vector space,  $[A, B] = I\Omega I^\top$ , stability is satisfied).*

- (1) *Suppose that there exists a subspace  $R' \subseteq R$  which is stable with respect to the maps  $v \mapsto vA$ ,  $v \mapsto vB$  and containing the image of the linear map*

$$\tilde{I} : \text{Hom}(\mathbb{C}^r, \mathbb{C}^{n-k}) \rightarrow R, \quad X \mapsto XI^\top.$$

*Then  $R' = R$ .*

- (2) *Let  $\alpha, \beta \in \text{Mat}((n-k) \times (n-k))$ ,  $[\alpha, \beta] = 0$  and let  $T_{A,\alpha}, T_{B,\beta} \in \text{End}(R)$  defined by*

$$v \mapsto vA - \alpha v, \quad v \mapsto vB - \beta v$$

*respectively. Then the identity*

$$\text{im}(T_{A,\alpha}) + \text{im}(\tilde{I}) + \text{im}(T_{B,\beta}\tilde{I}) + \text{im}(T_{B,\beta}^2\tilde{I}) + \cdots + \text{im}(T_{B,\beta}^{\dim(R)-1}\tilde{I}) = R$$

*holds.*

*Proof.* The first part is very easy. It is enough to prove that for a given  $R'$  as in the hypothesis, any matrix of the form

$$v = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v_i \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad v_i \in \mathbb{C}^k$$

belongs to  $R'$ . We will prove it only for  $i = 1$ , as the other cases are completely analogous.

The linear subspace  $S \subseteq \mathbb{C}^k$  given by vectors  $s$  such that  $\begin{pmatrix} s \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in R'$  must be necessarily the whole of  $\mathbb{C}^k$ , since

$$\begin{pmatrix} s \\ 0 \\ \vdots \\ 0 \end{pmatrix} A = \begin{pmatrix} sA \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in R',$$

we see that  $S$  is  $A$ -stable (and similarly,  $B$ -stable). Furthermore,

$$X_1 = \begin{pmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \text{Hom}(\mathbb{C}^r, \mathbb{C}^{n-k}) \implies \tilde{I}(X_1) = X_1 I^\top = \begin{pmatrix} x_1 I^\top \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in R',$$

so  $S \supseteq \text{im}(I)$ , and we are done thanks to the stability of the triple  $(A, B, I)$ .

To prove the second part, just apply the first to

$$R' = \text{im}(T_{A,\alpha}) + \text{im}(\tilde{I}) + \text{im}(T_{B,\beta}\tilde{I}) + \text{im}(T_{B,\beta}^2\tilde{I}) + \cdots + \text{im}(T_{B,\beta}^{\dim(R)-1}\tilde{I}).$$

By definition  $R'$  contains  $\text{im}(\tilde{I})$ , so we just need to prove it is  $A, B$ -stable in the above sense. We note that any  $v \in \text{im}(\tilde{I}) + \text{im}(T_{B,\beta}\tilde{I}) + \text{im}(T_{B,\beta}^2\tilde{I}) + \cdots + \text{im}(T_{B,\beta}^{\dim(R)-1}\tilde{I})$  can be rewritten as

$$v = X'_0 I^\top + X'_1 I^\top B + \cdots + X'_{k-1} I^\top B^{k-1},$$

because

$$T_{B,\beta}^m \tilde{I} X = X I^\top B^k - \binom{m}{2} \beta X I^\top B^{k-1} + \cdots - (1)^m \beta^m X I^\top.$$

So, we may write:

$$R' = \text{im}(T_{A,\alpha}) + \text{im}(\tilde{I}) + \text{im}(\tilde{B}\tilde{I}) + \text{im}(\tilde{B}^2\tilde{I}) + \cdots + \text{im}(\tilde{B}^{k-1}\tilde{I})$$

where  $\tilde{B}v = vB$ . Now we have:

$$\begin{aligned} \tilde{A}(T_{A,\alpha}(v)) &= T_{A,\alpha}(\tilde{A}v) \in R' \\ \tilde{A}(\tilde{B}^k\tilde{I}(X)) &= XI^\top B^k A = XI^\top B^k A - \alpha XI^\top B^k + \alpha XI^\top B^k = T_{A,\alpha}(XI^\top B^k) + \tilde{B}^k\tilde{I}(\alpha X) \in R' \\ \tilde{B}(T_{A,\alpha}(v)) &= vAB - \alpha vB = vBA + vI\Omega I^\top - \alpha vB = T_{A,\alpha}(vB) + \tilde{I}(vI\Omega) \in R' \\ \tilde{B}(\tilde{B}^k\tilde{I}(X)) &\in \text{Im}(\tilde{B}^{k+1}\tilde{I}) \subseteq R'. \end{aligned}$$

This guarantees  $A, B$ -stability of  $R'$ , and concludes the proof.  $\square$

We are now able to apply Lemma 5.8 to our ADHM quadruple

$$\left( \begin{pmatrix} A & 0 \\ a & \alpha \end{pmatrix}, \begin{pmatrix} B & 0 \\ b & \beta \end{pmatrix}, \begin{pmatrix} I \\ X \end{pmatrix}, \begin{pmatrix} \mathbb{1}_k & 0 \\ 0 & 0_{n-k} \end{pmatrix} \right)$$

in the following way. Thanks to Lemma 5.10, we can write

$$a = vA - \alpha v + X_0 I^\top + T_{B,\beta}(X_1 I^\top) + \cdots T_{B,\beta}^{k-1}(X_{k-1} I^\top),$$

and we can change the coordinates to eliminate the addend  $vA - \alpha v$ . We apply Lemma 5.8 to find a small deformation of  $a$ , written as  $a_t = a + v(t)$ , satisfying

$$(T_{B,\beta} - tId)v(t) = ta + Y(t)\Omega I^\top \implies (T_{B,\beta} - tId)a_t + T_{B,\beta}(a) = Y(t)\Omega I^\top$$

with  $Y(0) = 0$ , and so we obtain

$$\begin{aligned} a_t(B - tId) - \beta a_t - bA + \alpha b &= \\ = aB - \beta a - bA + \alpha b + Y(t)\Omega I^\top &= (X + Y(t))\Omega I^\top. \end{aligned}$$

In other words, the deformation

$$\left( \begin{pmatrix} A & 0 \\ a_t & \alpha \end{pmatrix}, \begin{pmatrix} B - tId & 0 \\ b & \beta \end{pmatrix}, \begin{pmatrix} I \\ X + Y(t) \end{pmatrix}, \begin{pmatrix} \mathbb{1}_k & 0 \\ 0 & 0_{n-k} \end{pmatrix} \right)$$

still sits in the space of symplectic ADHM data, and remains stable for small  $t$ . Obviously, the spectra of  $\beta$  and  $B - tId$  are disjoint for arbitrarily small nonzero values of  $t$ ; therefore, up to small deformations, we can indeed assume  $b = 0$ , by means of Lemma 5.9 together with the usual change of coordinates  $a \mapsto vA - \alpha v + a$ ,  $b \mapsto vB - \beta v + b$ .

So we can suppose without loss of generality that our quadruple is of the form

$$\left( \begin{pmatrix} A & 0 \\ a & \alpha \end{pmatrix}, \begin{pmatrix} B & 0 \\ 0 & \beta \end{pmatrix}, \begin{pmatrix} I \\ X \end{pmatrix}, \begin{pmatrix} \mathbb{1}_k & 0 \\ 0 & 0_{n-k} \end{pmatrix} \right).$$

Now we are only left to apply what we have learned from the rank 0 case, which is to play with nonderogatory matrices. First, we note that we can deform  $\alpha$  as  $\alpha_t = (1 - t)\alpha + t\eta$  for any matrix  $\eta$  commuting with  $\beta$ , and this does not harm the ADHM equations or the symmetries (this is why we made all the work to get  $b = 0$ , to make sure that no term of type  $b\alpha_t$  comes to ruin the party). This way, we can suppose that  $\alpha$  is nonderogatory up to small deformations,



and has no eigenvalues in common with  $A$  (it is enough to choose  $\eta$  nonderogatory and, if necessary, modify  $\alpha$  once again by adding small multiples of  $Id_{n-k}$  to slide the eigenvalues). By changing the coordinates as usual, we obtain a quadruple of type

$$\left( \begin{pmatrix} A & 0 \\ 0 & \alpha \end{pmatrix}, \begin{pmatrix} B & 0 \\ b & \beta \end{pmatrix}, \begin{pmatrix} I \\ X \end{pmatrix}, \begin{pmatrix} \mathbb{1}_k & 0 \\ 0 & 0_{n-k} \end{pmatrix} \right),$$

which is exactly as before except for a crucial detail:  $\alpha$  is nonderogatory. This means that by acting with a change of coordinates of type  $\begin{pmatrix} \mathbb{1}_k & 0 \\ 0 & g \end{pmatrix}$ , we can suppose that the commuting matrices  $\alpha$  and  $\beta$  are symmetric. Finally, let  $\chi$  be a symmetric matrix satisfying  $[\alpha, \chi] = X\Omega X^\top$ : it must exist, due to the cyclicity of  $\alpha$ .

Write down the final deformation

$$\left( \begin{pmatrix} A & 0 \\ 0 & \alpha \end{pmatrix}, \begin{pmatrix} B & tb^\top \\ b & \beta + t\chi \end{pmatrix}, \begin{pmatrix} I \\ X \end{pmatrix}, \begin{pmatrix} \mathbb{1}_k & 0 \\ 0 & t\mathbb{1}_{n-k} \end{pmatrix} \right).$$

Let us verify that this curve sits in the space of ADHM symplectic data:

$$\begin{aligned} & \begin{pmatrix} \mathbb{1}_k & 0 \\ 0 & t\mathbb{1}_{n-k} \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & \alpha \end{pmatrix} - \begin{pmatrix} A & 0 \\ 0 & \alpha \end{pmatrix}^\top \begin{pmatrix} \mathbb{1}_k & 0 \\ 0 & t\mathbb{1}_{n-k} \end{pmatrix} = \\ & \begin{pmatrix} A - A^\top & 0 \\ 0 & t\alpha - t\alpha^\top \end{pmatrix} = 0. \\ & \begin{pmatrix} \mathbb{1}_k & 0 \\ 0 & t\mathbb{1}_{n-k} \end{pmatrix} \begin{pmatrix} B & tb^\top \\ b & \beta + t\chi \end{pmatrix} - \begin{pmatrix} B & tb^\top \\ b & \beta + t\chi \end{pmatrix}^\top \begin{pmatrix} \mathbb{1}_k & 0 \\ 0 & t\mathbb{1}_{n-k} \end{pmatrix} = \\ & = \begin{pmatrix} B - B^\top & tb^\top - tb^\top \\ tb - tb & t\beta - t\beta^\top + t^2\chi - t^2\chi^\top \end{pmatrix} = 0. \end{aligned}$$

These give the  $G$ -symmetries. Recall that the ADHM equation for  $t = 0$  is given by  $\alpha b - bA = X\Omega I^\top$ , and that  $\Omega^\top = -\Omega$ . We verify the ADHM equation along the curve:

$$\begin{aligned} & \left[ \begin{pmatrix} A & 0 \\ 0 & \alpha \end{pmatrix}, \begin{pmatrix} B & tb^\top \\ b & \beta + t\chi \end{pmatrix} \right] - \begin{pmatrix} I \\ X \end{pmatrix} \Omega (I^\top \ X^\top) \begin{pmatrix} \mathbb{1}_k & 0 \\ 0 & t\mathbb{1}_{n-k} \end{pmatrix} = \\ & = \begin{pmatrix} [A, B] - I\Omega I^\top & tAb^\top - tb^\top\alpha - tI\Omega X^\top \\ \alpha b - bA - X\Omega I^\top & t[\alpha, \chi] - tX\Omega X^\top \end{pmatrix} = 0 \end{aligned}$$

This finishes the proof of the irreducibility, since  $G_t = \begin{pmatrix} \mathbb{1}_k & 0 \\ 0 & t\mathbb{1}_{n-k} \end{pmatrix}$  is invertible and so the general point of this curve lies in the locally free locus of the moduli space.

**Corollary 5.11.**  $\dim(\mathcal{M}_\Omega(r, n)) = \dim(\mathcal{M}_\Omega^{reg}(r, n)) = rn + 2n$ .

## 6. RELATIONS WITH UHLENBECK SPACES AND SINGULARITIES

The aim of this last section is twofold. First, we want to construct a proper birational map from the moduli space of framed symplectic sheaves into the space of symplectic ideal

instantons. The map will simply be the restriction of the so called Gieseker-to-Uhlenbeck map, defined on the moduli space of framed sheaves (we refer to [BMT] for the precise definition of this morphism). This provides a concrete example of the fact that Uhlenbeck spaces of type  $C$  can be obtained by generalized blow-downs of moduli spaces of symplectic sheaves, as suggested in [Bal, Ba2]. To this end, we will recall the definitions of Uhlenbeck spaces in a purely algebraic setting. In contrast with the classical case, the moduli space  $\mathcal{M}_\Omega(r, n)$  is singular in general. The second part of the section presents some results about its singular locus.

**6.1. Uhlenbeck spaces.** We recall the definition of Uhlenbeck spaces.

**Definition 6.1.** Let  $(r, n)$  be positive integers. We call *Uhlenbeck space* or *space of framed ideal instantons* the affine scheme defined by the categorical quotient

$$\mathcal{M}_0(r, n) := \mathbb{M}(r, n) // GL(V).$$

The following theorem summarizes some of the properties of this scheme. For details, see [Na, BFG].

**Theorem 6.2.** *The following statements hold:*

- (1)  $\mathcal{M}_0(r, n)$  is reduced and irreducible;
- (2) the open embedding  $\mathbb{M}^{sc}(r, n) \hookrightarrow \mathbb{M}(r, n)$  descends to an open embedding  $\mathcal{M}^{reg}(r, n) \rightarrow \mathcal{M}_0(r, n)$ , which is the smooth locus of  $\mathcal{M}_0(r, n)$ ;
- (3)  $\mathcal{M}_0(r, n)$  has a stratification into locally closed subsets of the form

$$\mathcal{M}_0(r, n) = \bigsqcup_{k=0}^n \mathcal{M}^{reg}(r, n-k) \times (\mathbb{A}^2)^{(k)},$$

where  $(\mathbb{A}^2)^{(k)}$  is the  $k$ -th symmetric power of the affine space  $\mathbb{P}^2 \setminus l$ ;

- (4) the open embedding  $\mathbb{M}^s(r, n) \rightarrow \mathbb{M}(r, n)$  descends to a projective morphism

$$\pi : \mathcal{M}(r, n) \rightarrow \mathcal{M}_0(r, n)$$

which is a resolution on singularities.

*Remark 6.3.* As a set-theoretic map,  $\pi$  has a very simple description. Let  $(E, a)$  be a framed sheaf of charge  $n$ ; the locally free sheaf  $E^{\vee\vee}$  inherits a framing  $a^{\vee\vee}$  from  $(E, a)$ , and sits in an exact sequence

$$0 \longrightarrow E \longrightarrow E^{\vee\vee} \longrightarrow C \longrightarrow 0$$

where  $C$  is a 0-dimensional sheaf supported away from  $l$ . Let  $Z(C)$  be the corresponding 0-cycle on  $\mathbb{A}^2$ . As

$$\text{length}(C) + c_2(E^{\vee\vee}) = n,$$

we obtain a point

$$\pi([E, a]) = ([E^{\vee\vee}, a^{\vee\vee}], Z(C)) \in \mathcal{M}_0(r, n).$$

In particular, we see that the restriction of  $\pi$  to the locally free locus  $\mathcal{M}^{reg}(r, n)$  induces an isomorphism onto the open subscheme  $\mathcal{M}^{reg}(r, n) \subseteq \mathcal{M}_0(r, n)$ .

It is possible to construct a symplectic variant of the Uhlenbeck space. Once again, let  $V \cong \mathbb{C}^n$  and  $W \cong \mathbb{C}^r$ . Fix a symplectic structure  $\Omega$  on  $W$  and a symmetric bilinear nondegenerate form  $1_V$  on  $V$ , and call  $End^+(V)$  the space of symmetric endomorphisms of  $V$ . Define the subspace of  $End^+(V)^{\oplus 2} \oplus Hom(W, V)$ :

$$\mathbb{X}(r, n) = \{(A, B, I) \mid [A, B] + I\Omega I^\top = 0\}$$

This space is naturally acted by the orthogonal group  $O(V)$ .

*Remark 6.4.* Let  $\tau : \mathbb{X}(r, n) \rightarrow \mathbb{M}_\Omega(r, n)$  be the embedding defined by

$$\tau(A, B, I) \rightarrow (A, B, I, 1_V).$$

The morphism  $\tau$  is equivariant with respect to the group homomorphism  $O(V) \rightarrow GL(V)$  defined by  $1_V$ . If we set  $\mathbb{X}^s : \tau^{-1}(\mathbb{M}_\Omega^s) = \tau^{-1}(\mathbb{M}_\Omega^{sc})$  (see Lemma 4.11), we obtain indeed an isomorphism of algebraic varieties

$$\mathbb{X}^s(r, n)/O(V) \rightarrow \mathbb{M}_\Omega^{sc}(r, n)/GL(V) (\cong \mathcal{M}_\Omega^{reg}(r, n)).$$

**Definition 6.5.** We define the symplectic Uhlenbeck space as the categorical quotient

$$\mathcal{M}_{0,\Omega}(r, n) = \mathbb{X}(r, n)//O(V).$$

We list some interesting properties of this affine scheme. For details, see [BFG, NS, Ch].

**Theorem 6.6.** *The following statements hold:*

- (1)  $\mathcal{M}_{0,\Omega}(r, n)$  is reduced and irreducible;
- (2) the open embedding  $\mathbb{X}^s(r, n) \hookrightarrow \mathbb{X}(r, n)$  descends to an open embedding  $\mathcal{M}_\Omega^{reg}(r, n) \rightarrow \mathcal{M}_{0,\Omega}(r, n)$ , which is the smooth locus of  $\mathcal{M}_{0,\Omega}(r, n)$ ;
- (3)  $\mathcal{M}_{0,\Omega}(r, n)$  has a stratification into locally closed subsets of the form

$$\mathcal{M}_{0,\Omega}(r, n) = \bigsqcup_{k=0}^n \mathcal{M}_\Omega^{reg}(r, n-k) \times (\mathbb{A}^2)^{(k)};$$

- (4) the composition

$$\mathbb{X}(r, n) \xrightarrow{\tau} \mathbb{M}_\Omega(r, n) \xrightarrow{\iota} \mathbb{M}(r, n)$$

is an equivariant closed embedding, inducing a closed embedding

$$\mathcal{M}_{0,\Omega}(r, n) \rightarrow \mathcal{M}_0(r, n)$$

which is compatible with the stratifications, meaning that for any integer  $k$

$$(\mathcal{M}^{reg}(r, n-k) \times (\mathbb{A}^2)^{(k)}) \cap \mathcal{M}_{0,\Omega}(r, n) = \mathcal{M}_\Omega^{reg}(r, n-k) \times (\mathbb{A}^2)^{(k)}$$

holds.

As a direct consequence of the discussion of the previous two sections, we find a relation between symplectic Uhlenbeck spaces and moduli of framed symplectic sheaves.

**Theorem 6.7.** *The moduli space  $\mathcal{M}_\Omega(r, n)$  is isomorphic to the strict transform of the closed subscheme  $\mathcal{M}_{0,\Omega}(r, n) \subseteq \mathcal{M}_0(r, n)$  under the resolution  $\pi$ .*

*Proof.* The maximal open subset of  $\mathcal{M}_0(r, n)$  over which  $\pi$  is an isomorphism is  $\mathcal{M}^{reg}(r, n)$ ; it follows by definition that the strict transform in the statement is defined as

$$\overline{\pi^{-1}(\mathcal{M}_{0,\Omega}(r, n) \cap \mathcal{M}^{reg}(r, n))} = \overline{\pi^{-1}(\mathcal{M}_\Omega^{reg}(r, n))} = \mathcal{M}_\Omega(r, n),$$

since  $\mathcal{M}_\Omega(r, n)$  is the smallest closed subscheme of  $\mathcal{M}(r, n)$  containing the locus of symplectic bundles, see Sect. 5.  $\square$

*Remark 6.8.* We note that  $\mathcal{M}_\Omega(r, n)$  also coincides with the total transform  $\pi^{-1}(\mathcal{M}_{0,\Omega}(r, n))$ , as its points are exactly the framed sheaves whose double dual is symplectic; indeed, if

$$\varphi' : E^{\vee\vee} \rightarrow E^{\vee\vee\vee} = E^\vee$$

is a “honest” symplectic form on the double dual, its composition with  $E \rightarrow E^{\vee\vee}$  endows  $E$  with a structure of framed symplectic sheaf.

## 6.2. Some remarks on the singularities of $\mathcal{M}_{\mathbb{P}^2,\Omega}(r, n)$ .

*Remark 6.9.*  $\mathcal{M}_{\mathbb{P}^2,\Omega}(r, 1)$  is smooth for any even  $r$ .

*Proof.* We know

$$\mathcal{M}_{\mathbb{P}^2,\Omega}(r, 1) \cong \mathbb{M}_\Omega^s(r, 1)/\mathbb{C}^*$$

where  $\mathbb{M}_\Omega^s(r, 1)$  is just the affine variety  $\mathbb{C}^2 \times (\mathbb{C}^r \setminus \{0\}) \times \mathbb{C}$ ; indeed, the symmetries and the ADHM equation are vacuous in this case. Since the  $\mathbb{C}^*$ -action is free on this space, no singularities can arise in the quotient.  $\square$

The case  $n = 1$  is special, since the singular locus of the Uhlenbeck space  $\mathcal{M}_{0,\Omega}(r, 1)$  is smooth (therefore, a single blow-up is enough to resolve the singularities). In fact, singularities appear in  $\mathcal{M}_\Omega(r, n)$  as we take  $n > 1$ . In the next example, we draw our attention to the case  $n = 2$ .

**Example 6.10.** Let  $I : W \rightarrow \mathbb{C}^2$  be a linear map, and consider the ADHM quadruple  $\xi = (0, 0, I, 0) \in \mathbb{M}_\Omega(r, 2)$ . This configuration will be stable if we choose  $I$  to be surjective. Let us compute the dimension of  $T_\xi \mathbb{M}_\Omega^s(r, 1)$ : using the description of the tangent space in the proof of 4.12, we have

$$T_\xi \mathbb{M}_\Omega^s(r, 1) = \{(X_A, X_B, X_I, X_G) \mid I\Omega I^\top X_G = 0\}.$$

If we choose  $I$  so that  $I\Omega I^\top$  is not invertible (it is enough to require the rows of the matrix  $I$  to span an isotropic subspace of  $W$ , and this can be done while keeping  $I$  surjective if  $r \geq 4$ ), we obtain

$$\dim(T_\xi \mathbb{M}_\Omega^s(r, 1)) > 2 \cdot 2^2 + 2r = 8 + 2r.$$

The dimension of  $\mathbb{M}_\Omega^s(r, 1)$  is exactly  $8 + 2r$ , so we found a singular point.

Unfortunately, Cor. 3.13 has no easy translation into the language of ADHM data, but we can still say something on the nonsingular locus of  $\mathcal{M}_\Omega(r, n)$ .

*Remark 6.11.* Suppose that a point  $x = [(E, \alpha, \varphi)] \in \mathcal{M}_\Omega(r, n)$  satisfies the hypothesis of Cor. 3.13. Let  $\iota(x) \in \mathcal{M}(r, n)$  be the corresponding framed sheaf. We have an exact sequence of vector spaces

$$0 \longrightarrow T_x \mathcal{M}_\Omega(r, n) \longrightarrow T_{\iota(x)} \mathcal{M}(r, n) \longrightarrow \text{Ext}_{\mathcal{O}_X}^1(\Lambda^2 E, \mathcal{O}_X(-1)) \longrightarrow 0$$

This forces

$$\text{ext}^1(\Lambda^2 E, \mathcal{O}_X(-1)) = 2nr - rn - 2n = rn - 2n.$$

Now, by Serre duality, we have

$$\text{Ext}_{\mathcal{O}}^1(\Lambda^2 E, \mathcal{O}_X(-1)) \cong H^1(\mathbb{P}^2, \Lambda^2 E(-2)).$$

Let  $E'$  be a framed bundle with the same numerical invariants of  $E$ . The bundle  $\Lambda^2 E'$  is a framed bundle as well, and we can compute:

$$c_2(E') = H^1(\mathbb{P}^2, \Lambda^2 E'(-2)) = -\chi(\Lambda^2 E'(-2)) = nr - 2n,$$

see [Na, Sect. 2.1]. We deduce

$$-\chi(\Lambda^2 E(-2)) = nr - 2n$$

since this quantity only depends on the numerical invariants. Let  $T \subseteq \Lambda^2 E$  be the torsion subsheaf.  $T$  is concentrated on points since  $E$  is locally free away from a codimension two locus. Call  $F = \Lambda^2 E/T$ ;  $F$  is again a framed sheaf, and from the long exact sequence in cohomology coming from the short exact sequence

$$0 \longrightarrow T \longrightarrow \Lambda^2 E(-2) \longrightarrow F(-2) \longrightarrow 0$$

we get

$$\chi(\Lambda^2 E(-2)) = \text{length}(T) - h^1(\Lambda^2 E(-2)).$$

In order to obtain  $\chi(\Lambda^2 E(-2)) = \text{ext}^1(\Lambda^2 E, \mathcal{O}_X(-1))$  as desired, we are forced to ask  $T = 0$ . Now, the torsion of the sheaf  $E^{\otimes 2}$  vanishes if and only if  $E$  is locally free (see [Au, Thm. 4]); we deduce that the hypothesis of 3.13 can hold if and only if the torsion is concentrated in the summand  $S^2 E$ .

The previous discussion implies that no symplectic sheaf  $E$  whose residue  $C = E^{\vee\vee}/E$  is a skyscraper sheaf  $\mathbb{C}_x$  (i.e.  $E$  has a unique singular point with multiplicity 1) can satisfy the hypothesis. We sketch a proof of this fact. First, we can suppose  $x = (0, 0) \in \mathbb{A}^2$ . Passing to stalks on  $x$ , we can write  $E_x$  as the kernel of a quotient

$$\mathcal{O}^{\oplus r} \rightarrow \mathbb{C}_{(0,0)}.$$

Denote by  $m$  the ideal  $(x, y) \subseteq \mathbb{C}[x, y]$ . The element

$$x \wedge y \in \Lambda^2 m$$

can be proved very easily to be nonzero and torsion; from this we get that  $\Lambda^2 E_x$  has torsion, and this implies that  $\Lambda^2 E$  has torsion as well.

Anyway, the following proposition will show that 3.13 does not give necessary conditions for a point to be smooth.

**Proposition 6.12.** *Let  $\xi \in \mathcal{M}_\Omega(r, n)$  be represented by an ADHM quadruple  $(A, B, I, G)$  such that either  $A$  or  $B$  is a nonderogatory endomorphism. Then  $\xi$  is a nonsingular point.*

Indeed, using the constructions in Section 5 it is immediate to produce examples of symplectic sheaves whose residue has length equal to 1 and which are represented by quadruples with  $A$  nonderogatory. We now prove the proposition.

*Proof.* We use once again the ADHM-theoretic description of the tangent space as exposed in the proof of Lemma 4.12, i.e. we think of  $T_{(A, B, I, G)} \mathbb{M}_\Omega^s(r, n)$  as the kernel of the the Jacobian matrix

$$\begin{pmatrix} X_A \\ X_B \\ X_I \\ X_G \end{pmatrix} \mapsto \begin{pmatrix} G_- - \_^\top G & 0 & 0 & \_A - A^\top \_ \\ 0 & G_- - \_^\top G & 0 & \_B - B^\top \_ \\ \_ [ , B] & \_ [ A, \_] & (I \Omega \_^\top + \_ \Omega I^\top) G & \_ I \Omega I^\top \_ \end{pmatrix} \begin{pmatrix} X_A \\ X_B \\ X_I \\ X_G \end{pmatrix}$$

$$\begin{pmatrix} X_A \\ X_B \\ X_I \\ X_G \end{pmatrix} \in \text{End}(V)^{\oplus 2} \oplus \text{Hom}(W, V) \oplus \text{Hom}(S^2 V, \mathbb{C}).$$

Suppose  $A$  is nonderogatory. We change coordinates to make  $A$  symmetric, so that we can write  $\_A - A^\top \_ = -[A, \_]$ . The ciclicity of  $A$  guarantees the surjectivity of the map  $[A, \_]$  as a morphism on the space of symmetric matrices with values in the space of skew-symmetric matrices. If we think  $[A, \_]$  as a linear map on the space of general square matrices, its rank is  $n^2 - n$ . We deduce immediately that the rank of the Jacobian is greater or equal to  $\frac{3}{2}n(n-1)$ , hence

$$\begin{aligned} \dim(T_{(A, B, I, G)} \mathbb{M}_\Omega^s(r, n)) &\leq 2n^2 + nr + \frac{1}{2}n(n+1) - \frac{3}{2}n(n-1) = \\ &= n^2 + nr + 2n = \dim(\mathcal{M}_\Omega(r, n)) + n^2 \leq \dim(T_{(A, B, I, G)} \mathbb{M}_\Omega^s(r, n)). \end{aligned}$$

This implies that the equation

$$\dim T_\xi(\mathcal{M}_\Omega(r, n)) = \dim(\mathcal{M}_\Omega(r, n))$$

holds, as required.  $\square$

The locus of sheaves  $[(A, B, I, G)]$  with  $A$  nonderogatory is a nonempty open subscheme of codimension greater or equal than 1. Indeed, its complement is contained in the locus cut out

by the discriminant of the characteristic polynomial of  $A$  (call it  $\Delta_A$ ; it is a  $GL(V)$ -invariant function). Any matrix  $A$  whose discriminant is nonzero will be nonderogatory, since in this case  $A$  does not admit double eigenvalues. In particular, the singular locus is contained in the intersection

$$\{\Delta_A = 0\} \cap \{\Delta_B = 0\} \cap \{\det(G) = 0\}.$$

This intersection cannot be of codimension 1 unless these divisors all share some irreducible components, and they do not (to see this, it is again enough to play with deformations as in Section 5). We conclude:

**Corollary 6.13.**  $\mathcal{M}_\Omega(r, n)$  is nonsingular in codimension 2.

## REFERENCES

- [ADHM] M. Atiyah, N. Hitchin, V. Drinfel'd, Y. Manin, *Construction of instantons*, Phys. Lett. A , 65(3), 185-187 (1978)
- [Au] M. Auslander, *Modules over unramified local rings*, Illinois J. Math. Volume 5, Issue 4, 631-647 (1961)
- [Bal] V. Balaji, *Principal bundles on projective varieties and the Donaldson-Uhlenbeck compactification*, J. Differential Geom. 76.3, 351-398 (2007)
- [Ba1] V. Baranovsky, *Moduli of sheaves on surfaces and action of the oscillator algebra*, Journal of Differential Geometry 55.2, 193-227 (2000)
- [Ba2] V. Baranovsky, *Uhlenbeck compactification as a functor*, Int. Mathematics Research Notices 23, 12678-12712 (2015)
- [BFG] A. Braverman, M. Finkelberg, D. Gaitsgory, *Uhlenbeck spaces via affine Lie algebras*, The unity of mathematics, 17-135 (2006)
- [BM] U. Bruzzo, D. Markushevich, *Moduli of framed sheaves on projective surfaces*, Documenta Math. 16, 399-410 (2011)
- [BMT] U. Bruzzo, D. Markushevich, A. Tikhomirov, *Uhlenbeck-Donaldson compactification for framed sheaves on projective surfaces*, Math. Z. 275, 1073-1093 (2013)
- [Ch] J. Choy, *Moduli spaces of framed symplectic and orthogonal bundles on  $\mathbb{P}^2$  and the K-theoretic Nekrasov partition functions*, PhD Thesis, Kyoto University (2015)
- [Do] S. Donaldson, *Instantons and geometric invariant theory*, Comm. Math. Phys. 93(4), 453-460 (1984)
- [GD] A. Grothendieck, J. Dieudonné, *Éléments de géométrie algébrique: II. Étude globale élémentaire de quelques classes de morphismes*, Publ. Math. IHES 8, p. 5-222 (1966)
- [GS1] T.L. Gomez, I. Sols, *Stable tensors and moduli space of orthogonal sheaves*, arXiv:math/0103150v4 (2002)
- [GS2] T.L. Gomez, I. Sols, *Moduli space of principal sheaves over projective varieties*, Annals of Mathematics 161, 1037-1092 (2005)
- [He] A. A. Henni, *Monads for torsion-free sheaves on multi-blow-ups of the projective plane*, Int. J. Math. 25, 450008, 42 pages (2014)
- [HL] D. Huybrechts, M. Lehn, *Framed modules and their moduli*, Internat. J. Math. 6, 297-324 (1995)
- [JMW] M. Jardim, S. Marchesi and A. Wissdorf, *Moduli of autodual instanton bundles*, arXiv:1401.6635 (2014)
- [Mc] J. McCleary, *User's Guide to Spectral Sequences*, Cambridge Studies in Advanced Mathematics 58 (2nd ed.) (2001)
- [Na] H. Nakajima, *Lectures on Hilbert schemes of points on surfaces*, American Mathematical Soc., University Lecture Series No. 18 (1999)

- [NS] N. Nekrasov, S. Shadchin, *ABCD of instantons*, Communications in Mathematical Physics 252.1, 359–391(2004)
- [OSS] C. Okonek, M. Schneider, H. Spindler, *Vector bundles on complex projective spaces*, Progress in Math. 3, Birkhäuser (1980)
- [TZ] O. Taussky, H. Zassenhaus, *On the similarity transformation between a matrix and its transpose*, Pacific J. Math. Volume 9, Number 3, 893–896 (1959)